The How and Why [illegible] Variable
CALCULUS

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The How and Why of One Variable Calculus

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The How and Why of One Variable Calculus

To my parents

Preface

Who is this book is for?

This book is meant as a textbook for an honours course in Calculus, and is aimed at first year students beginning studies at the university. The preparation assumed is high school level Mathematics. Any arguments not met before in high school (for example, geometric arguments à la Euclid) can be picked up along the way or simply skipped without any loss of continuity. This book may also be used as supplementary reading in a traditional methods-based Calculus course or as a textbook for a course meant to bridge the gap between Calculus and Real Analysis.

How should the student read the book?

Students reading the book should not feel obliged to study every proof at the first reading. It is more important to understand the theorems well, to see how they are used, and why they are interesting, than to spend all the time on proofs. So, while reading the book, one may wish, after reading the theorem statement, to first study the examples and solve a few relevant exercises, before returning to read the proof of that theorem.

The exercises are an integral part of studying this book. They are a combination of purely drill ones (meant for practising Calculus methods), and those meant to clarify the meanings of the definitions, theorems, and even to facilitate the goal of developing 'mathematical maturity'. The student should feel free to skip exercises that seem particularly challenging at the first instance, and return back to them now and again. Although detailed solutions are provided, the student should not be tempted to consult the given solution too soon. In the learning process leading to developing understanding, it is much better to think about the exercise (even if one does not find the answer oneself!), rather than look at the provided solution in order to understand how to solve it. In other words, it is the *struggle* to solve the exercise that turns out to be more important than the mere *knowledge* of the solution. After all, given a *new* problem, it will be the struggle that pays off, and not the knowledge of the solution of the (now irrelevant) *old* exercise! So the student should absolutely not feel discouraged if he or she doesn't manage to solve an exercise problem. Some of the exercises that are more abstract/technical/challenging as compared to the other exercises are indicated with an asterisk symbol $(*)$.

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I would like to thank Sara Maad Sasane (Lund University) for going through the entire manuscript, pointing out typos and mistakes, and offering insightful suggestions and comments. Thanks are also due to Lassi Paunonen (Tampere University) and Raymond Mortini (University of Lorraine-Metz) for many useful comments. A few pedagogical ideas in this book stem from some of the references listed at the end of this book. This applies also to the exercises. References are given in the section on notes at the end of the chapters, but no claim to originality is made in case there is a missing reference. The figures in this book have been created using xfig, Maple, and MATLAB. Finally, it is a pleasure to thank the editors and staff at Wiley, especially Debbie Jupe, Heather Kay, and Prachi Sinha Sahay. Thanks are also due to the project manager, Sangeetha Parthasarathy, for cheerfully and patiently overseeing the typesetting of the book.

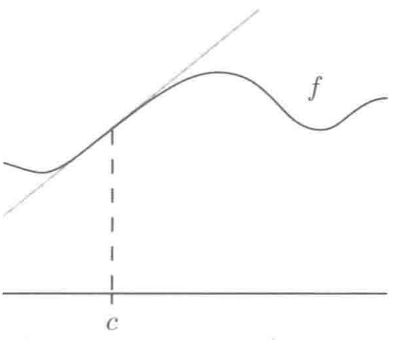
Amol Sasane
London, 2014.

Introduction

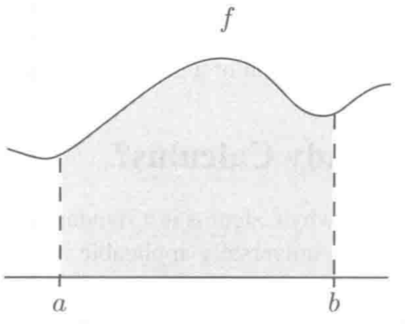
What is Calculus?

Calculus is a branch of mathematics in which the focus is on two main things: given a real-valued function of a real variable, what is the rate of change of the function at a point (Differentiation), and what is the area under the graph of the function over an interval (Integration).

Differentiation and **Integration**



What is the steepness/slope of f at the point c?



What is the area under the graph of f over an interval from a to b?

Differentiation	**Integration**
Differentiation is concerned with velocities, accelerations, curvatures, etc. These are rates of change of function values and are defined *locally*.	Integration is concerned with areas, volumes, average values, etc. These take into account the totality of function values, and are *not* defined locally.

We will see later on that the rate of change of f at c is defined by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

and what matters is not what the function is doing *far away* from the point c, but rather the manner in which the function behaves in the *vicinity* of c. This is what we mean when we say that 'differentiation is a local concept'. On the other hand, we will learn that for nice functions, the area will be given by an expression that looks like

$$\int_a^b f(x)dx = \lim_{N \to \infty} \sum_{n=0}^{N} f\left(a + \frac{b-a}{N}n\right)\frac{b-a}{N},$$

and we see that in the above process, the values of the function over the entire interval from a to b *do matter*. In this sense integration is a 'non-local' or 'global' process.

Thus it seems that in Calculus, there are these two quite *different* topics of study. However there is a remarkable fact, known as that *Fundamental Theorem of Calculus*, which creates a bridge between these seemingly different worlds: it says, roughly speaking that the processes of differentiation and integration are inverses of each other:

$$\int_a^b f'(x)dx = f(b) - f(a) \quad \text{and} \quad \frac{d}{dx}\int_a^x f(\xi)d\xi = f(x).$$

This interaction between differentiation and integration provides a powerful body of understanding and calculational technique, called 'Calculus'. Problems that would be otherwise computationally difficult can be solved mechanically using a few simple Calculus rules, and without the exertion of a great deal of penetrating thought.

Why study Calculus?

The reason why Calculus is a standard component of all scientific undergraduate education is because it is universally applicable in Physics, Engineering, Biology, Economics, and so on. Here are a few very simple examples:

(1) What is the escape velocity of a rocket on the surface of the Earth?

(2) If a hole of radius 1 cm is drilled along a diametrical axis in a solid sphere of radius 2 cm, then what is the volume of the body left over?

(3) If a strain of bacteria grows at a rate proportional to the amount present, and if the population doubles in an hour, then what is the population of bacteria at any time t?

(4) If the manufacturing cost of x lamps is given by $C(x) = 2700 - 100x$, and the revenue function is given by $R(x) = x - 0.03x^2$, then what is the number of lamps maximising the profit?

We will primarily be concerned with developing and understanding the tools of Calculus, but now and then in the exercises and examples chosen, we will consider a few toy models from various application areas to illustrate how the techniques of Calculus have universal applications.

What will we learn in this book?

This book is divided into six chapters, listed below.

(1) The real numbers.
(2) Sequences.
(3) Continuity.
(4) Differentiation.
(5) Integration.
(6) Series.

This covers the core component of a *single/one* variable Calculus course, where the basic object of study is a real-valued function of *one* real variable, and one studies the themes of differentiation and integration for such functions. On the other hand, in *multi/several* variable Calculus, the basic object of study is an $\mathbb{R}^m$-valued/vector valued function of several variables, and the themes of differentiation and integration for such functions. This book does not cover this latter subject.

We refrain from giving a brief gist of the contents of each of the chapters, since it won't make much sense to the novice at this stage, but instead we appeal to whatever previous exposure the student might have had in high school regarding these concepts. We will of course study each of these topics from scratch. We make one pertinent point though in the paragraph below.

A discussion of Calculus needs an ample supply of examples, which are typically through considering specific functions one meets in applications. The simplest among these illustrative functions are the algebraic functions, but it would be monotonous to just consider these. Much more interesting things happen with the so-called elementary transcendental functions such as the logarithm, exponential function, trigonometric functions, and so on. A rigorous definition of these unfortunately needs the very tools of Calculus that are being developed in this course. It would be a shame, however, if such rich examples centered around these functions have to wait till a rigorous treatment has been done. So we adopt a dual approach: we *will* choose to illustrate our definitions/theorems with these functions, and *not exclude* these functions from our preliminary discussion, hoping that the student has *some* exposure to the definitions (at whatever intuitive/rigour level) and properties of these transcendental functions. Later on, when the time is right (Chapter 5), we will give the precise mathematical definitions of these functions and prove the very properties that were accepted on faith in the initial parts of this book. This dual approach adopted by us has the advantage of not depriving the student of the nice illustrations of the results provided by these functions, and of preparing the student for the actual treatment of these functions later on. In any case, if the student meets a very unfamiliar property or manipulation involving these functions in the initial part of this book, it is safe to simply skip the relevant part and revisit it after Chapter 5 has been read.

How did Calculus arise?

Some preliminary ideas of Calculus are said to date back to as much as 2000 years ago when Archimedes determined areas using the Method of Exhaustion; see the following discussion and Figure 1.

The development of Differential and Integral Calculus is largely attributed to Newton (1642–1727) and Leibniz (1646–1716), and the foundations of the subject continued to be investigated into the 19th century, among others by Cauchy, Bolzano, Riemann, Weierstrass, Lebesgue, and so on.

We end this introduction with making a few remarks about the 'Method of Exhaustion', which besides treating this historical milestone in the development of Calculus, will also provide some motivation to begin our journey into Calculus with a study of the real numbers.

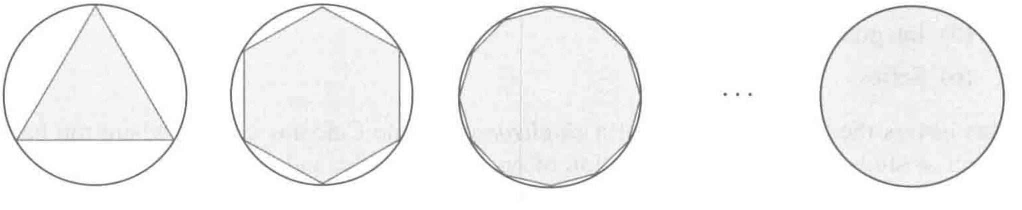


Figure 1. Determination of the area of a circle using the Method of Exhaustion.

In Figure 1, it is clear that what we are doing is trying to obtain the area of a circle by inscribing polygons inside it, each time doubling the number of sides, hence 'exhausting' more and more of the circular area. The idea is then that if A is the area of the circle we seek, and a_n is the area of the polygon at the nth step, then for large n, a_n approximates A. As we have that $a_1 \leq a_2 \leq a_3 \leq \cdots$, and since a_n misses A by smaller and smaller amounts as n increases, we expect that A should the 'smallest' number exceeding the numbers $a_1, a_2, a_3, \cdots$. Does such a number always exist?

Obviously, one can question the validity of this heuristic approach to solving the problem. The objections are for example:

(1) We did not really define what we mean by the area enclosed.

(2) We are not sure about what properties of numbers we are allowed to use. For example, we seem to be needing the fact that 'if we have an increasing sequence of numbers, all of which are less than a certain number[1], then there is a smallest number which is bigger than each of the numbers $a_1, a_2, a_3, \cdots$'. Is this property true for rational numbers?

Such questions might seem frivolous to a scientist who is just interested in 'real world applications'. But such a sloppy attitude can lead to trouble. Indeed, some work done in the 16th to the 18th century relying on a mixture of deductive reasoning and intuition, involving vaguely defined terms, was later shown to be *incorrect*. To give the student a quick example of how things might easily go wrong, one might naively, but incorrectly, guess that the answer to question (2) above is yes. This prompts the question of whether there is a bigger set of numbers than the rational numbers for which the property happens to be true? The answer is yes, and this is the **real number system** $\mathbb{R}$.

Thus a thorough treatment of Calculus must start with a careful study of the number system in which the action of Calculus takes place, and this is the real number system $\mathbb{R}$, where our journey begins!

[1] imagine a square circumscribing the circle: then each of the numbers $a_1, a_2, a_3, \cdots$ are all less than the area of the square

Preliminary notation

$A := B$ or $B =: A$	A is defined to be B; A is defined by B
$\forall$	for all; for every
$\exists$	there exists
$\neg S$	negation of the statement S; it is not the case that S
$a \in A$	the element a belongs to the set A
$\emptyset$	the empty set containing no elements
$A \subset B$	A is a subset of B
$A \subsetneq B$	A is a subset of B, but is not equal to B
$A \backslash B$	the set of elements of A that do not belong to B
$A \cap B$	intersection of the sets A and B
$\bigcap_{i \in I} A_i$	intersection of the sets A_i, $i \in I$
$A \cup B$	union of the sets A and B
$\bigcup_{i \in I} A_i$	union of the sets A_i, $i \in I$
$A_1 \times \cdots \times A_n$	Cartesian product of the sets $A_1, \cdots, A_n$; $\{(a_1, \cdots, a_n) : a_1 \in A_1, \cdots, a_n \in A_n\}$

Contents