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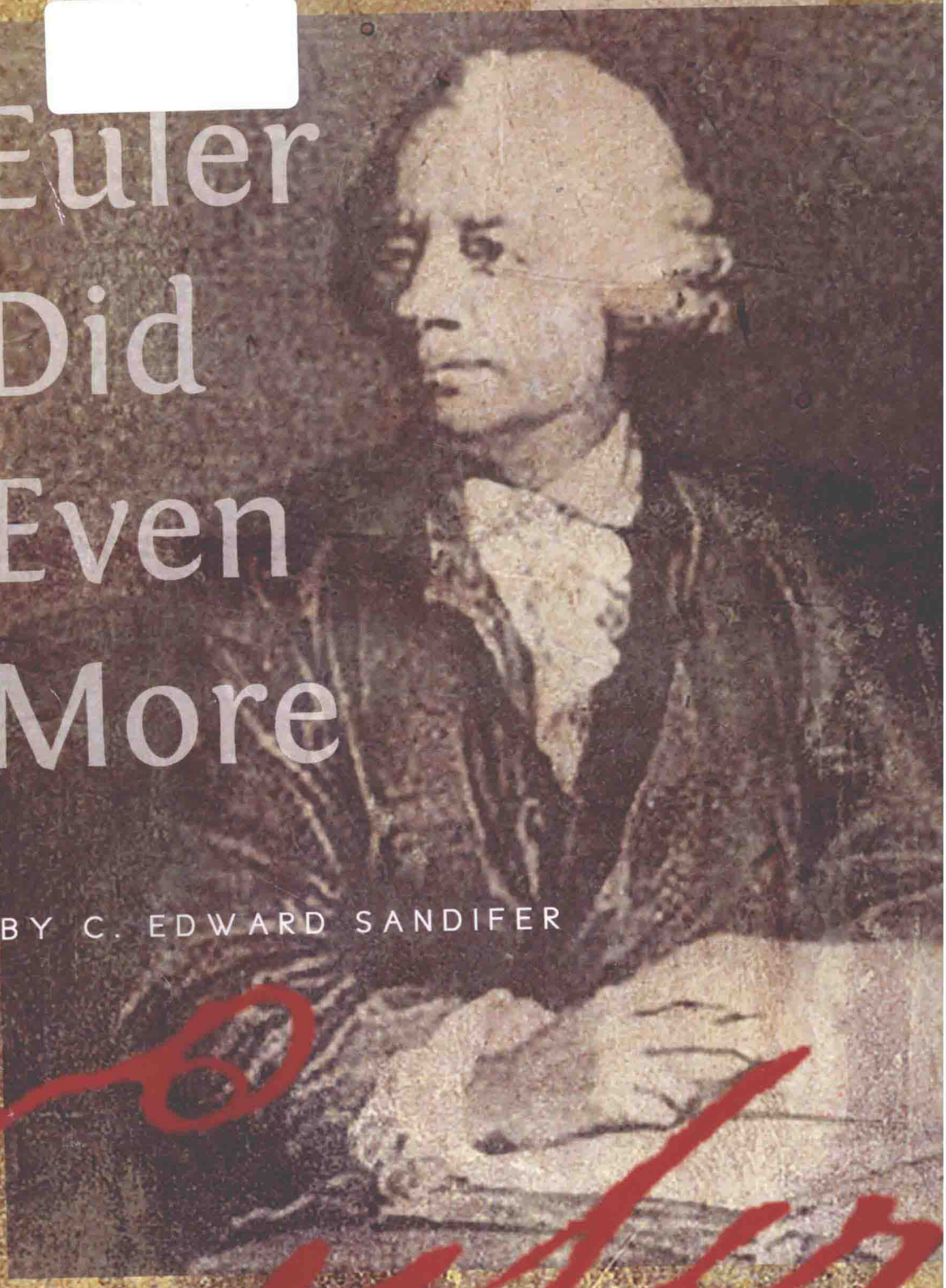

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How Euler Did Even More

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Preface

In the first years of this century, the mathematics community was gearing up for Leonhard Euler's tercentenary. I see the official kick-off as being a special session sponsored by the Mathematical Association of America (MAA) at the 2001 Joint Mathematics Meeting called "Mathematics in the Age of Euler," organized by William Dunham and V. Frederick Rickey. At the same meeting, The Euler Society was first conceived of; it would come into existence the next year and begin holding annual meetings devoted to Euler studies. As 2007 approached, the MAA planned a special five-volume book series called *The MAA Tercentenary Euler Celebration*. The MAA invited the Euler Society to make the 2007 MathFest in San Jose, CA, a joint meeting of the two societies. MathFest was the climax of the tercentenary celebrations, featuring a plenary address from Bill Dunham, the official release of the last two volumes of the *Tercentenary* book series, and a presentation of a copy of the entire series to the Swiss consul, who traveled from the consulate in San Francisco to be a part of the celebration.

Ed Sandifer was the driving force behind much of this activity. I first met Ed in 1999, when he spoke at my university in the Pohle Colloquium on the History of Mathematics on—who else?—Leonhard Euler. Even though the big anniversary was still eight years off, Ed had already begun travelling the country, sharing the excitement of reading Euler in the original, and encouraging faculty and students alike to translate Euler's many papers and books into English.

Two years later, at the Joint Mathematics Meeting in New Orleans, Ed discussed the formation of an academic society devoted to Euler, his work, and his intellectual community, with Ronald Calinger and John Glaus. And so the Euler Society was born. Ed served as the secretary of the society from its inception until long after the tercentenary celebrations. As secretary, he tirelessly recruited members, promoted the society's annual meetings and made proceedings available in electronic form.

At about the same time that he helped found The Euler Society, Ed began work on a book that would eventually appear as the first volume in the MAA *Tercentenary* book series, titled *The Early Mathematics of Leonhard Euler*. It is the only comprehensive survey of Euler's groundbreaking mathematical papers from the early years of his career, before his move to Berlin in 1741.

In late 2003, Ed began writing a series of monthly columns that appeared on MAA Online called *How Euler Did It*. Each column was a self-contained exploration of one of Euler's theorems, or some particular facet of his work, illustrated with figures and excerpts from Euler's books and papers. The first 40 of these columns, through February 2007, were collected in the third volume of the MAA *Tercentenary* book series, appropriately titled *How Euler Did It*.

As the tercentenary drew nearer, Ed and I began editing two volumes of papers on Euler, both of which appeared in 2007. The first of these—*Leonard Euler: Life, Work and Legacy*—was published by Elsevier. The second collection, for which Lawrence D'Antonio also served as editor, became the fifth and final volume of the MAA *Tercentenary* book series, called *Euler at 300: An Appreciation*. Of the five volumes in the series, Ed was responsible for two and a third!

After the tercentenary celebrations, Ed devoted some of his seemingly boundless energy to other projects, such as our annotated translation of Cauchy's *Cours d'analyse*, published by Springer in 2009. He also continued his monthly online column for the MAA. By the summer of 2009, he had made plans to wind down *How Euler Did It*, and possibly start a new on-line column for the MAA. He determined that his swan song would be a two-part column in January and February of 2010, on Euler as a teacher. This would mean a full three years' worth of additional material, which he hoped would be collected in a second volume of *How Euler Did It*.

Sadly, Ed suffered a stroke in August of 2009. It happened just days after he attended MathFest in Portland, OR, where he gave the keynote address "Prove it Again, Sam" at the Opening Banquet and pitched a new book proposal to Birkhäuser. Those who know Ed, and especially those of us who ran with him, understand the irony that someone who was so ridiculously healthy and fit (he completed 37 consecutive Boston Marathons, beginning in 1973) should have suffered from blockage of the carotid artery. The stroke was severe. Ed was initially unable to read, write or talk and he suffered significant loss of mobility on his right side. He took to the various therapies that were prescribed for him with the same energy and dedication that he applies to everything he does. Within a year, he was able to read again. He can now type with his left hand and his speech has gradually improved over the years so that it is now, as Ed himself might say, "good, but not great." Ed's many friends and colleagues were happy to see him attend the 2012 Joint Mathematics Meeting in Boston, MA, and the 2013 MathFest in Hartford, CT.

Ed had already submitted his August 2009 *How Euler Did It* column at the time of the stroke. He also had his final two-part column in a reasonably good draft form, which I assisted him in editing in December of that year. But what to do about the autumn months of 2009? With Ed's consent, I submitted a couple of guest columns to the MAA that fall. Dominic Klyve, a member of the Euler Society's executive committee, also contributed a guest piece, so the flow of columns was almost without noticeable interruption. In 2002, when Dominic was in his first year of graduate studies, Ed visited Dartmouth (his alma mater) and gave a talk on Euler. Dominic and his fellow graduate student Lee Stemkoski were inspired by Ed's presentation and attended the first meeting of the Euler Society later that year. Not long thereafter, the two of them created the Euler Archive, a vital online resource for Euler studies. It's particularly fitting that someone whom Ed so deeply inspired was able to contribute a column in Ed's time of need.

The final 35 *How Euler Did It* columns are all collected in this book. 32 of them were written by Ed. They are lightly edited versions of the columns as they actually appeared on MAA Online between March 2007 and February 2010. The order of the articles as presented here is thematic, not chronological. As with the first volume of columns, we have assigned most of them into sections on Geometry, Number Theory, Combinatorics and Analysis. (Euler being Euler, the Analysis section is the largest.) There is also a section on Applied Mathematics, which opens with a column about Euler's study of saws, that also includes some of Ed's general observations about Euler's applied work. The final section, which we have dubbed "Euleriana," consists of papers devoted, at least in part, to Euler himself, including misattributions, mistakes, and a light-hearted column about "Euler and the Pirates."

I had the pleasure of working closely with Ed during the time that he wrote these columns. More than once at the end of a long day working on some other project, we would relax over a beer and he would tell me of some new Eulerian gem that he had uncovered in his reading, outlining his plans to feature it in an upcoming column. It's a great pleasure to know that all of Ed's columns will soon be in print, and a great privilege to have worked with him in making this book a reality.

Rob Bradley
Adelphi University
September 16, 2013

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Part I

Geometry