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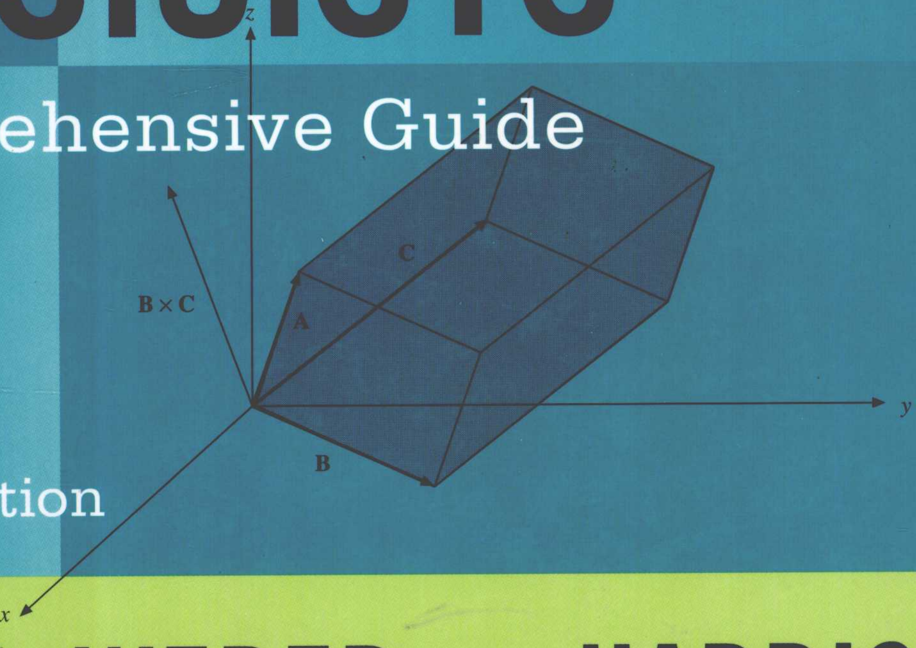
MATHEMATICAL METHODS for PHYSICISTS

A Comprehensive Guide



Seventh Edition

ARFKEN, WEBER, AND HARRIS

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A Comprehensive Guide

SEVENTH EDITION

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PREFACE

This, the seventh edition of *Mathematical Methods for Physicists*, maintains the tradition set by the six previous editions and continues to have as its objective the presentation of all the mathematical methods that aspiring scientists and engineers are likely to encounter as students and beginning researchers. While the organization of this edition differs in some respects from that of its predecessors, the presentation style remains the same: Proofs are sketched for almost all the mathematical relations introduced in the book, and they are accompanied by examples that illustrate how the mathematics applies to real-world physics problems. Large numbers of exercises provide opportunities for the student to develop skill in the use of the mathematical concepts and also show a wide variety of contexts in which the mathematics is of practical use in physics.

As in the previous editions, the mathematical proofs are not what a mathematician would consider rigorous, but they nevertheless convey the essence of the ideas involved, and also provide some understanding of the conditions and limitations associated with the relationships under study. No attempt has been made to maximize generality or minimize the conditions necessary to establish the mathematical formulas, but in general the reader is warned of limitations that are likely to be relevant to use of the mathematics in physics contexts.

TO THE STUDENT

The mathematics presented in this book is of no use if it cannot be applied with some skill, and the development of that skill cannot be acquired passively, e.g., by simply reading the text and understanding what is written, or even by listening attentively to presentations by your instructor. Your passive understanding needs to be supplemented by experience in using the concepts, in deciding how to convert expressions into useful forms, and in developing strategies for solving problems. A considerable body of background knowledge needs to be built up so as to have relevant mathematical tools at hand and to gain experience in their use. This can only happen through the solving of problems, and it is for this reason that the text includes nearly 1400 exercises, many with answers (but not methods of solution). If you are using this book for self-study, or if your instructor does not assign a considerable number of problems, you would be well advised to work on the exercises until you are able to solve a reasonable fraction of them.

This book can help you to learn about mathematical methods that are important in physics, as well as serve as a reference throughout and beyond your time as a student. It has been updated to make it relevant for many years to come.

WHAT'S NEW

This seventh edition is a substantial and detailed revision of its predecessor; every word of the text has been examined and its appropriacy and that of its placement has been considered. The main features of the revision are: (1) An improved order of topics so as to reduce the need to use concepts before they have been presented and discussed. (2) An introductory chapter containing material that well-prepared students might be presumed to know and which will be relied on (without much comment) in later chapters, thereby reducing redundancy in the text; this organizational feature also permits students with weaker backgrounds to get themselves ready for the rest of the book. (3) A strengthened presentation of topics whose importance and relevance has increased in recent years; in this category are the chapters on vector spaces, Green's functions, and angular momentum, and the inclusion of the dilogarithm among the special functions treated. (4) More detailed discussion of complex integration to enable the development of increased skill in using this extremely important tool. (5) Improvement in the correlation of exercises with the exposition in the text, and the addition of 271 new exercises where they were deemed needed. (6) Addition of a few steps to derivations that students found difficult to follow. We do not subscribe to the precept that "advanced" means "compressed" or "difficult." Wherever the need has been recognized, material has been rewritten to enhance clarity and ease of understanding.

In order to accommodate new and expanded features, it was necessary to remove or reduce in emphasis some topics with significant constituencies. For the most part, the material thereby deleted remains available to instructors and their students by virtue of its inclusion in the on-line supplementary material for this text. On-line only are chapters on Mathieu functions, on nonlinear methods and chaos, and a new chapter on periodic systems. These are complete and newly revised chapters, with examples and exercises, and are fully ready for use by students and their instuctors. Because there seems to be a significant population of instructors who wish to use material on infinite series in much the same organizational pattern as in the sixth edition, that material (largely the same as in the print edition, but not all in one place) has been collected into an on-line infinite series chapter that provides this material in a single unit. The on-line material can be accessed at www.elsevierdirect.com.

Pathways through the Material

This book contains more material than an instructor can expect to cover, even in a two-semester course. The material not used for instruction remains available for reference purposes or when needed for specific projects. For use with less fully prepared students, a typical semester course might use Chapters 1 to 3, maybe part of Chapter 4, certainly Chapters 5 to 7, and at least part of Chapter 11. A standard graduate one-semester course might have the material in Chapters 1 to 3 as prerequisite, would cover at least part of Chapter 4, all of Chapters 5 through 9, Chapter 11, and as much of Chapters 12 through 16 and/or 18 as time permits. A full-year course at the graduate level might supplement the foregoing with several additional chapters, almost certainly including Chapter 20 (and Chapter 19 if not already familiar to the students), with the actual choice dependent on the institution's overall graduate curriculum. Once Chapters 1 to 3, 5 to 9, and 11 have been covered or their contents are known to the students, most selections from the remaining chapters should be reasonably accessible to students. It would be wise, however, to include Chapters 15 and 16 if Chapter 17 is selected.

Acknowledgments

This seventh edition has benefited from the advice and help of many people; valuable advice was provided both by anonymous reviewers and from interaction with students at the University of Utah. At Elsevier, we received substantial assistance from our Acquisitions Editor Patricia Osborn and from Editorial Project Manager Kathryn Morrissey; production was overseen skillfully by Publishing Services Manager Jeff Freeland. FEH gratefully acknowledges the support and encouragement of his friend and partner Sharon Carlson. Without her, he might not have had the energy and sense of purpose needed to help bring this project to a timely fruition.

CONTENTS

CHAPTER 1

MATHEMATICAL PRELIMINARIES

This introductory chapter surveys a number of mathematical techniques that are needed throughout the book. Some of the topics (e.g., complex variables) are treated in more detail in later chapters, and the short survey of special functions in this chapter is supplemented by extensive later discussion of those of particular importance in physics (e.g., Bessel functions). A later chapter on miscellaneous mathematical topics deals with material requiring more background than is assumed at this point. The reader may note that the Additional Readings at the end of this chapter include a number of general references on mathematical methods, some of which are more advanced or comprehensive than the material to be found in this book.

1.1 INFINITE SERIES

Perhaps the most widely used technique in the physicist's toolbox is the use of **infinite series** (i.e., sums consisting formally of an infinite number of terms) to represent functions, to bring them to forms facilitating further analysis, or even as a prelude to numerical evaluation. The acquisition of skill in creating and manipulating series expansions is therefore an absolutely essential part of the training of one who seeks competence in the mathematical methods of physics, and it is therefore the first topic in this text. An important part of this skill set is the ability to recognize the functions represented by commonly encountered expansions, and it is also of importance to understand issues related to the convergence of infinite series.

Fundamental Concepts

The usual way of assigning a meaning to the sum of an infinite number of terms is by introducing the notion of partial sums. If we have an infinite sequence of terms $u_1, u_2, u_3, u_4, u_5, \ldots$, we define the ith partial sum as

$$s_i = \sum_{n=1}^{i} u_n. \tag{1.1}$$

This is a finite summation and offers no difficulties. If the partial sums s_i converge to a finite limit as $i \to \infty$,

$$\lim_{i\to\infty} s_i = S, \tag{1.2}$$

the infinite series $\sum_{n=1}^{\infty} u_n$ is said to be **convergent** and to have the value S. Note that we **define** the infinite series as equal to S and that a necessary condition for convergence to a limit is that $\lim_{n\to\infty} u_n = 0$. This condition, however, is not sufficient to guarantee convergence.

Sometimes it is convenient to apply the condition in Eq. (1.2) in a form called the **Cauchy criterion**, namely that for each $\varepsilon > 0$ there is a fixed number N such that $|s_j - s_i| < \varepsilon$ for all i and j greater than N. This means that the partial sums must cluster together as we move far out in the sequence.

Some series **diverge**, meaning that the sequence of partial sums approaches $\pm\infty$; others may have partial sums that oscillate between two values, as for example,

$$\sum_{n=1}^{\infty} u_n = 1 - 1 + 1 - 1 + 1 - \cdots - (-1)^n + \cdots .$$

This series does not converge to a limit, and can be called **oscillatory**. Often the term *divergent* is extended to include oscillatory series as well. It is important to be able to determine whether, or under what conditions, a series we would like to use is convergent.

Example 1.1.1 THE GEOMETRIC SERIES

The geometric series, starting with $u_0 = 1$ and with a ratio of successive terms $r = u_{n+1}/u_n$, has the form

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} + \cdots .$$

Its nth partial sum s_n (that of the first n terms) is[1]

$$s_n = \frac{1 - r^n}{1 - r}. \tag{1.3}$$

Restricting attention to $|r| < 1$, so that for large n, r^n approaches zero, and s_n possesses the limit

$$\lim_{n\to\infty} s_n = \frac{1}{1-r}, \tag{1.4}$$

[1] Multiply and divide $s_n = \sum_{m=0}^{n-1} r^m$ by $1 - r$.

showing that for $|r| < 1$, the geometric series converges. It clearly diverges (or is oscillatory) for $|r| \geq 1$, as the individual terms do not then approach zero at large n. ■

Example 1.1.2 THE HARMONIC SERIES

As a second and more involved example, we consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots . \tag{1.5}$$

The terms approach zero for large n, i.e., $\lim_{n\to\infty} 1/n = 0$, but this is not sufficient to guarantee convergence. If we group the terms (without changing their order) as

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots ,$$

each pair of parentheses encloses p terms of the form

$$\frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{p+p} > \frac{p}{2p} = \frac{1}{2}.$$

Forming partial sums by adding the parenthetical groups one by one, we obtain

$$s_1 = 1, \quad s_2 = \frac{3}{2}, \quad s_3 > \frac{4}{2}, \quad s_4 > \frac{5}{2}, \ldots, \quad s_n > \frac{n+1}{2},$$

and we are forced to the conclusion that the harmonic series diverges.

Although the harmonic series diverges, its partial sums have relevance among other places in number theory, where $H_n = \sum_{m=1}^{n} m^{-1}$ are sometimes referred to as **harmonic numbers**. ■

We now turn to a more detailed study of the convergence and divergence of series, considering here series of positive terms. Series with terms of both signs are treated later.

Comparison Test

If term by term a series of terms u_n satisfies $0 \leq u_n \leq a_n$, where the a_n form a convergent series, then the series $\sum_n u_n$ is also convergent. Letting s_i and s_j be partial sums of the u series, with $j > i$, the difference $s_j - s_i$ is $\sum_{n=i+1}^{j} u_n$, and this is smaller than the corresponding quantity for the a series, thereby proving convergence. A similar argument shows that if term by term a series of terms v_n satisfies $0 \leq b_n \leq v_n$, where the b_n form a divergent series, then $\sum_n v_n$ is also divergent.

For the convergent series a_n we already have the geometric series, whereas the harmonic series will serve as the divergent comparison series b_n. As other series are identified as either convergent or divergent, they may also be used as the known series for comparison tests.

Example 1.1.3 A Divergent Series

Test $\sum_{n=1}^{\infty} n^{-p}$, $p = 0.999$, for convergence. Since $n^{-0.999} > n^{-1}$ and $b_n = n^{-1}$ forms the divergent harmonic series, the comparison test shows that $\sum_n n^{-0.999}$ is divergent. Generalizing, $\sum_n n^{-p}$ is seen to be divergent for all $p \le 1$. ■

Cauchy Root Test

If $(a_n)^{1/n} \le r < 1$ for all sufficiently large n, with r independent of n, then $\sum_n a_n$ is convergent. If $(a_n)^{1/n} \ge 1$ for all sufficiently large n, then $\sum_n a_n$ is divergent.

The language of this test emphasizes an important point: The convergence or divergence of a series depends entirely on what happens for large n. Relative to convergence, it is the behavior in the large-n limit that matters.

The first part of this test is verified easily by raising $(a_n)^{1/n}$ to the nth power. We get

$$a_n \le r^n < 1.$$

Since r^n is just the nth term in a convergent geometric series, $\sum_n a_n$ is convergent by the comparison test. Conversely, if $(a_n)^{1/n} \ge 1$, then $a_n \ge 1$ and the series must diverge. This root test is particularly useful in establishing the properties of power series (Section 1.2).

D'Alembert (or Cauchy) Ratio Test

If $a_{n+1}/a_n \le r < 1$ for all sufficiently large n and r is independent of n, then $\sum_n a_n$ is convergent. If $a_{n+1}/a_n \ge 1$ for all sufficiently large n, then $\sum_n a_n$ is divergent.

This test is established by direct comparison with the geometric series $(1 + r + r^2 + \cdots)$. In the second part, $a_{n+1} \ge a_n$ and divergence should be reasonably obvious. Although not quite as sensitive as the Cauchy root test, this D'Alembert ratio test is one of the easiest to apply and is widely used. An alternate statement of the ratio test is in the form of a limit: If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \begin{cases} < 1, & \text{convergence,} \\ > 1, & \text{divergence,} \\ = 1, & \text{indeterminate.} \end{cases} \tag{1.6}$$

Because of this final indeterminate possibility, the ratio test is likely to fail at crucial points, and more delicate, sensitive tests then become necessary. The alert reader may wonder how this indeterminacy arose. Actually it was concealed in the first statement, $a_{n+1}/a_n \le r < 1$. We might encounter $a_{n+1}/a_n < 1$ for all **finite** n but be unable to choose an $r < 1$ **and independent of n** such that $a_{n+1}/a_n \le r$ for all sufficiently large n. An example is provided by the harmonic series, for which

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} < 1.$$

Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1,$$

no fixed ratio $r < 1$ exists and the test fails.

Example 1.1.4 D'ALEMBERT RATIO TEST

Test $\sum_n n/2^n$ for convergence. Applying the ratio test,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{1}{2}\,\frac{n+1}{n}.$$

Since

$$\frac{a_{n+1}}{a_n} \le \frac{3}{4} \quad \text{for } n \ge 2,$$

we have convergence. ■

Cauchy (or Maclaurin) Integral Test

This is another sort of comparison test, in which we compare a series with an integral. Geometrically, we compare the area of a series of unit-width rectangles with the area under a curve.

Let $f(x)$ be a continuous, **monotonic decreasing function** in which $f(n) = a_n$. Then $\sum_n a_n$ converges if $\int_1^\infty f(x)dx$ is finite and diverges if the integral is infinite. The ith partial sum is

$$s_i = \sum_{n=1}^{i} a_n = \sum_{n=1}^{i} f(n).$$

But, because $f(x)$ is monotonic decreasing, see Fig. 1.1(a),

$$s_i \ge \int_1^{i+1} f(x)dx.$$

On the other hand, as shown in Fig. 1.1(b),

$$s_i - a_1 \le \int_1^{i} f(x)dx.$$

Taking the limit as $i \to \infty$, we have

$$\int_1^\infty f(x)dx \le \sum_{n=1}^{\infty} a_n \le \int_1^\infty f(x)dx + a_1. \tag{1.7}$$

Hence the infinite series converges or diverges as the corresponding integral converges or diverges.

This integral test is particularly useful in setting upper and lower bounds on the remainder of a series after some number of initial terms have been summed. That is,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n, \tag{1.8}$$