A FOUNDATIONAL STUDY

IN THE

PEDAGOGY OF ARITHMETIC

BY

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JERSEY CITY, N. J.

New York
THE MACMILLAN COMPANY
1914

Set up and electrotyped Published October, 1914

PREFACE.

The problem of arithmetical teaching has been attacked in the past from many avenues of approach. Several interesting treatises, written from almost every point of view except from the insights gained by careful first-hand study of cases and experimental trial of adults and children, have appeared. While these have been suggestive and valuable according to their lights, manifestly they are makeshifts awaiting more thoroughgoing and systematic attempts to render available to teachers and students of education the scholarly investigations that have been going on for the past decade or more and that have been entombed for the most part in the libraries of our universities. Many of these need popularizing and to an extent rewriting, since, unfortunately, our best investigators are not always our best expositors.

If the author has succeeded in summarizing and organizing to some extent these widely scattered studies into the genesis, psychology, statistics, and didactics of number in a manner to make them readable and interesting as well as more concisely contributory to the kind of material which he feels must be the subject-matter of the prolegomena to any future arithmetical didactic, he will feel duly rewarded.

In all of this work as well as in the original work of part two, heartfelt acknowledgment of the helpfulness and guidance of Assistant Professor Paul R. Radosavljevich, of the faculty of the School of Pedagogy of New York University, is hereby made; also to Professor Robert McDougall and especially to Dean Balliet of this School for encouragement and counsel. Thanks

must also be given to the teachers of P. S. No. 27, Jersey City, for cheerful assistance in many of the little attractive but extremely valuable phases of the work of preparation; and to Professor S. A. Courtis, of the Detroit Home and Day School, for the use made of his tests and of some of his published as well as unpublished discussions.

INTRODUCTORY NOTE

The following treatise is based on a thesis on the psychology and pedagogy of arithmetic prepared by the author in the department of education of the graduate school of New York University and accepted by the faculty of that school in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

The first part of the book constitutes a very complete résumé of the extensive experimental literature on this subject and makes it readily accessible to the general student. It will serve equally well as a convenient guide to the special student who desires to make a first-hand study of it.

The second part forms the author's own research, and is an important contribution both because of its positive results and as a scholarly critique of the technique employed by other investigators.

The experimental method will do for pedagogy in the future what it has done so effectively for psychology during the last half century, by placing it on a more scientific basis and eliminating the element of speculation from the study of problems which lend themselves readily to experimental methods of investigation.

The definite results already attained, although relatively meager, are yet of such general interest and importance that we cannot afford to disregard them in making school programs and determining methods of teaching. The author of this book has therefore rendered an important service, besides making an original contribution of much interest, in making these results accessible, in convenient form, to the classroom teacher as well as to the busy superintendent and principal of schools.

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INTRODUCTION.

A study in the pedagogy of arithmetic will include at least the following lines of inquiry: (a) the origin and development of number concepts in primitive men and in children (anthropogenetic studies): (b) the mental functioning of adults capable of introspection and description with reference to the processes involved (psychological studies): (c) the objective study of abilities and efficiency and their economical development with a view to the possible discovery of relationships, causes and hindrances (statistical studies); (d) the objective study of the school child's apprehension of number (experimental didactics). In addition to these four main lines of inquiry and, indeed, involved in them to a greater or less extent, it will be desirable to consider somewhat such topics as the hygiene of arithmetic (normal and healthful learning); arithmetical prodigies (geniuses, but primitive in their arithmetical processes); transfer (how far does an ability developed to deal with one set of materials pass over to the dealing with another set); ideational types (arithmetical working-types of imagery). Upon the basis of such studies only, can be founded a scientific curriculum and a scientific method.

"No pedagogue has anywhere even attempted to sum up the copious but scattered anthropological, experimental, pedagogical, psychogenetic and phylogenetic resources now available" [in arithmetic] (G. Stanley Hall).[1] In the following pages an attempt, though far from exhaustive, will be made to give this summary.

Our purpose then is (1) to review at some length and in a

[1] 61, p. 393.

measure criticize, supplement and organize the recent literature bearing on the lines of inquiry mentioned above, and (2) to give an account of a series of experiments tried by the author in order to study at first hand (a) the arithmetical abilities of certain school children, (b) the problem of the school child's concepts of number.

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A REVIEW OF REPRESENTATIVE STUDY OF NUMBER.

1. Genetic Studies.[1]

(a) *Primitive Men.*

The origin and development of the number sense in primitive man has been studied in considerable detail by L. L. Conant (1) and W J McGee (2). Of course, primitive men have long since turned to dust; so that the method of procedure must be to investigate the conditions in existing peoples who are still in varying stages of savage culture. The anthropological assumption is that the course of mental progress is approximately uniform and is by *natural* means, so that the growth stage of a people may be determined by studying the mentality of other peoples developed to a corresponding stage.

It might be supposed that the development of the number sense could only proceed *pari passu* with the development of number names, and that the limits of the latter would set the limitations of the former; or again that the convenience for counters of the fingers and ten toes would give rise, very early in human history, to decimal or vigesimal number systems. But a study of the beginnings of number systems shows "that (1) the origin of number names is at the bottom of the scale of human development, that is to say, comes late, (2) primeval man does not cognize quinary and decimal systems, (3) does not use his toes, fingers, etc., as mechanical adjuncts to nascent notation" (2, p. 654). Many primitive peoples count by fingers and hands, sometimes with the addition of toes and feet, and thereby fix quinary, decimal and vigesimal systems; but the burden of evidence is that these are far from primeval.

The paucity of number names among savage tribes is illustrated by the following examples cited by McGee (2). Some Australian

[1] This review of Genetic Studies appeared originally in Educational Foundations, 1914, XXV, pp. 5–10.

tribes count laboriously up to 2, 3, 4, or 6, sometimes doubling 2 to make 4 or 3 to make 6 and in other ways revealing a quasi-binary system; certain Brazilian tribes are described as counting only to 2, 3, 4, usually with an additional name for many; the Tasmanians counted commonly to 2, sometimes to 4, and were able to reach 5 by the addition of one to the limital number (four and one more). But it should be noted that this paucity of number names did not set the limit to numerical capacity. Most tribesmen reveal the germ of notation in the use of sticks, notches, etc. By means of these and of gesture language they symbolized numerical values beyond the limits of the number names. In the savage mind, and (as we shall see) in the child's mind, the series-idea as an abstraction precedes the naming. Modern names are the product of gradual evolution motivated by a felt necessity to express the series-idea already present. Our savage ancestors found it a tremendous task to reach a very meager number expression; that their conception of number far outran their ability to express is shown by much anthropological evidence, Conant (1), Phillips (5); e.g., the original inhabitants of Victoria had no numerals above two, yet they counted and even recorded the phases of the moon.

The foundational study of McGee (2) bids us consider as fundamental to an understanding of the number systems of the earliest men that these men were beyond everything egoists and mystics with strange and (to us) mysterious ways of orientation. Binary, quaternary and senary systems become to them subconsciously ternary, quinary and septenary respectively. Quaternary does not remain purely quaternary, but by the addition of a vague unity (the ego, from which all things procecd) becomes quinary, the system of fours becomes in thought as well as in graphic representation a system of fives, sixes becomes sevens, and these by a process of *augmentation* form the bases of the larger numbers which belong to the respective systems.

An examination of some of the mechanical symbolisms will serve to make the method clear. The four-five system is represented by + or ∴∵. (Note that the point of intersection of the lines of the cross, or the middle dot represents the unity that is always counted in by the primeval man, himself, the It.) This symbol is raised from 4 + 1 to 8 + 1 by adding a line or dot to each of the four cardinal extremities, thus  or  "The mystical middle is persistent and can be counted but once howsoever the value be augmented" (2, p. 662). Again the 8 + 1 is raised to 12 + 1 by the addition of dots, thus ; or still higher,  representing 20 + 1, etc.; or by the development of the "meander,"  representing 16 + 1, etc. The augmentation may proceed thus indefinitely:

Four-five system—$\frac{4+1}{5}$ $\frac{8+1}{9}$ $\frac{12+1}{13}$ $\frac{16+1}{17}$ $\frac{20+1}{21}$ $\frac{24+1}{25}$ $\frac{28+1}{29}$ $\frac{32+1}{33}$, etc.

The six-seven system is produced by the superposition of a binary system on the quaternary system. "It is more complicated and modified through the difficulty of depicting tridimensional relations on a bidimensional surface. Among the Pueblo peoples this is overcome by bisecting two of the quadrants, thus  but mechanical tendency operates to produce the regular figure  (2, p. 662). From the first figure augmentation produces  representing 12 + 1, which later is modified into the hexagram 

Six-seven system— $\frac{6+1}{7}$ $\frac{12+1}{13}$ $\frac{18+1}{19}$ $\frac{24+1}{25}$ $\frac{30+1}{31}$ $\frac{36+1}{37}$ $\frac{42+1}{43}$, etc.

The binary-ternary system, from which the quatern-quinary system is derived by augmentation, is lacking in graphic representation. It was produced in a manner similar to the others.

Two-three system— $\frac{2+1}{3}$ $\frac{4+1}{5}$ $\frac{6+1}{7}$ $\frac{8+1}{9}$ $\frac{10+1}{11}$ $\frac{12+1}{13}$ $\frac{14+1}{15}$.

It will be observed that some of the numbers occur in more than one system; these were deemed of unusual mystical significance. "These number systems are distinct from Aryan arithmetic, both in motive and mechanism. They are devices for divination, for binding the real world to the supernal, and it is only later or in an ancillary way that they are prostituted to practical uses; yet by reason of the extraordinary potency imputed to them they dominate thought and action in the culture stages to which they belong and profoundly affect the course of intellectual development. The base of the system is a measure of the intellectual capacity normal to the culture stage to which it belongs" (2, p. 660).

The investigation shows that the binary-ternary system is the earliest; higher in the scale of human development are the quatern-quinary and senary-septenary systems. It may seem strange that having hit upon a concept of five in the quatern-quinary thenceforth the development of the convenient decimal system should not be rapid. But the *definite* quinary concept from which the step to ten would have been easy was missed by many primitive tribes, although it was possessed by the ancestors of the Arabs with their decimal system, by the Mexicans who had a vigesimal system, and a few others; and we find in many lands a distinct development of the senary-septenary system instead. Finger counting, then, is a comparatively late development, for if it had existed early, it would have led to the early rise of the decimal system. Direct observation of savage tribes in the earliest stages of culture confirm this absence of finger counting, for many of them are found to be unable to count their fingers without the use of other symbols.

It would be of more anthropological than mathematical interest to trace the influence of the almacabala described above on the social and fiducial systems of primeval man and their relation to his ways of orientating himself. The dualistic (binary) conception of the cosmos existing among the earliest men (and further back among animals), the danger side in front, the safety side in the rear, merges into the antithesis of sea and land; the sea full of horrors, the land the haven of safety. Still later the strand of the shore stretching on either hand, as the sea is faced, characterizes the quaternary stage of culture. More important from our point of view is the *law of augmentation* which lies at the basis of the source of certain vestigial features still persisting in Aryan culture.

The three number systems described have all left vestiges, some of which persist even to modern times among civilized peoples. They may be traced to peoples still living in the lower culture stages, thus serving to establish the course of the development of number concepts and to throw light upon the numeration of our savage ancestors.

A few examples cited by McGee follow: The 6–7 system survives as the bridge connecting almacabala and mathematics. In the graphic form it became the Pythagorean hexagon of two superposed triangles, the hexagram of Brianchon (Paracelsus), the subrational hexagram of Pascal, etc. The astrologic seven retarded acceptance of the discovery of the eighth planet, Neptune. In the numerical form six, and more especially seven, play large rôles in the classical and sacred literature revived during the Elizabethan period. Nine (from the 2–3 and 4–5 systems) survives in the Muses, nine lives of the cat, effeminacy of the tailor; it even survived in the school books of the early part of the century in the more curious than useful arithmetic process of casting out nines; thirteen in all three systems is still the messenger of evil in the minds of many; seventh son of a seventh son needs no training for medical craft, nor seventh daughter of seventh daughter as seeress.

It seems, then, that we have here the historical presentation of the beginning of mathematics. The archaic method was "to use integral numbers as tokens of extra-natural potencies rather than as symbols for natural values; to combine them by a simple rule tending to develop into algorismic processes; and to represent the numerical combinations by mechanical devices tending to develop into geometrical forms; the system being characterized by the method of reckoning from an ill-defined unity counted but once in each combination" (2, p. 664), "so that in all cases the exoterically mystical number carried an esoteric complement in the form of a simple unity reflecting the egoistic personality or subjectivity of the thinker" (2, p. 656).

The quinary, decimal and vigesimal systems set forth by Conant (1) with their accompaniment of counting by fingers and hands, sometimes with toes and feet, mark later stages of culture and form transitional steps to our modern system. As mankind progressed, counting was divorced from its supernal connections and became motivated more and more by practical ends of more precise adjustment to the environment. But for a long, long time man remained in the mere counting stage objectifying his series by objects and marks for the most part (number names being but a meager possession); it is only when the savage becomes a thinking human being that number in the mathematical sense can be grasped by him. At this point "mere reckoning ceases and arithmetic begins" (1, p. 73).

To go back into our animal ancestry to search for earlier beginnings is interesting but futile. It is certain that the highest animals and the lowest savages have in common mathematical ability of a certain (or uncertain) sort which enables them to distinguish differences between small groups, and in some cases apparently to determine absolute number. The classical instance of the crow which could tell that one was absent from a group of not more than four men (simultaneous presentation); the nightingale which could tell when it had finished its meal of three worms given one at a time (successive presentation); the parent wasp which supplies its young with a food supply of five to twenty-four victims, no more, no less, according to its species; these are cases in point. Did these animals count or was their apprehension merely quantitative? Whatever the answer it applies equally well to the animal or the savage, since the objective product is the same in both. It would seem that we have here a capacity ultimate in its nature, with its roots lying far back in prehuman evolution, which because of its ultimate nature must, as Conant says, be left in the region of pure speculation.

(b) *Children.*

All will agree, perhaps, that progress beyond mere rudimentary quantitative discrimination depends upon *the ability to count.* There seems to be little difference of opinion concerning this among observers, experimenters, psychologists and philosophers (Lay, Meumann, Phillips, Messenger, Dewey). However intuitive and instantaneous the apprehension of number may seem to the mature mind, psychogenetic analysis shows it to be generated by slow degrees through counting. The child, then, must and does count. Does the child count spontaneously, instinctively, or must he await the development of a power, however low, of abstraction and generalization?

This problem has been quite thoroughly investigated by Phillips (5). He experimented on kindergarten children, consulted primary teachers, and collected data from 616 persons by means of a questionnaire, 235 men, 319 women, 62 neutral; 72 per cent of these teachers, 40 per cent having taught over five years.

Phillips found the following facts which called for interpretation.

Children, before they have learned the number names, are observed to follow a succession of stimuli or to create a succession long before there is any conscious idea of number. They will repeat a series of sounds, as the strokes of the clock, throw down a given number of blocks time and again. A boy under two rolled one after the other ten mud balls down an incline, marking one each time until each contained a little cross. Tallying with the fingers or toes, nodding the head, rhythmical articulations of various sorts, etc., are the characteristic responses in this stage.

Having learned the number names as a series of auditory symbols mechanically associated with one another (this he does very soon through imitation before he learns to read or to write), he uses them indiscriminately and without reference to objecta of any kind. Thirty-three out of thirty-nine children in a kindergarten counted without reference to objects. Primary teachers consulted say that for some time children count in this manner, and, if an attempt is made to apply the counting, the series of names constantly tends to run far ahead of the objects. Most children are found to learn the names independently of the objects. "I placed before little Willie little sticks and told him to touch each stick as he counted; he was just as apt to say six or any other number when he touched three as to say it correctly" (5, p. 262). Children show a passion for counting. They eagerly seek the names, catch them readily and use them delightedly. They will use the few names they know at first, repeating them over and over again, for a series of indefinite length. They recognize three or four objects at first as individuals, calling the fourth one four even when set aside by itself. They change the order of the number names, the particular articulatory response being often a matter of indifference in such cases as elsewhere in their counting. To sum up negatively these results in a word: embryonic counting is not a response to outer stimuli; it is not the result of the observation of sensible things. The positive correlate to this is, counting at first is a motor response to an *inner* series. But what is the nature of this inner series and how is it established?

Phillips (5) says that it is a consciousness of succession that has resulted from a long experience of successions in consciousness. Moreover, the changes are naïvely rhythmical. This rhythmical subjectivity is an ultimate fact, a possession common to all, though not the same in all. Our minds are attuned to rhythmical responses but they do not all play the same tune. Experiments performed upon the rhythmical sense show an indifference point, a length of interval between the beats of a metronome, say, more easily and accurately judged than any other above or below, but it varies with individuals and with the different senses. Series of metronome beats tend to group themselves into rhythmic multiples. This is a matter of common experience in listening to the tick of a clock. Children inquire why some ticks are longer than others. This synchronization, as I understand it, is a category of the subjective ordering of experiences which we cannot help if we would. The experiences themselves begin at birth. Some of these are rhythmical *per se* and may serve at times as symbols for an internal rhythm, as the pulse, respiration, walking, etc. Changes in consciousness are continually taking place produced by the varying impressions from all the senses. The earliest and most rudimentary form of knowledge is a knowledge of a series of changes. "All the experimental work substantiates James's thought that number is primarily strokes of attention." "The tactile sense very early produces an endless series of changes in consciousness which soon become vaguely recognized as distinct both in time and space.