NAVIER-STOKES EQUATIONS AND RELATED NONLINEAR PROBLEMS

EDITORS:
H. AMANN, G. P. GALDI,
K. PILECKAS AND V. A. SOLONNIKOV

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Navier-Stokes Equations and Related Nonlinear Problems

FOREWORD

This volume contains the proceedings of the Sixth International Conference on Navier–Stokes Equations and Related Nonlinear Problems (NSEC6), held in Palanga, Lithuania, on May 22-29, 1997. This meeting continued the series of conferences which, since 1992, were held regularly in different countries.

The Palanga meeting brought together 66 researchers from all over the world and provided a forum for presenting the newest results, for discussing mathematical questions of common interest, as well as open problems. While the emphasis was on the mathematical foundation of fluid dynamics, the conference attracted related contributions from Nonlinear and Numerical Analysis as well.

This volume is a collection its 25 articles selected from invited lectures and contributed papers. The main topics covered include: Incompressible Fluids Described by the Navier–Stokes Equations; Compressible Fluid; Non-Newtonian Fluids; Free Boundary Problems; Equations from Thermo- and Magnetohydrodynamics; Asymptotic Analysis; Stability; Related Problems of Nonlinear and Numerical Analysis. The papers are either original results or updated surveys of recent developments, giving directions for future research.

The conference would not have been possible without the financial support from the Open Society Fund-Lithuania, Lithuanian Sciencies and Studies Foundation, Scientific Publishing House TEV, and the Institute of Mathematics and Informatics of Lithuania. We are greatly indebted to all the sponsors. We wish also to express our deepest gratitude to the local organizers: Professors M. Sapagovas, V. Statulevičius, R. Čiegis and S. Rutkauskas. We are also indebted to Professor V. Būda for his invaluable technical support during the conference, and we thank the TEV editors for preparation of contributions for publishing.

Last but not least, we thank all participants for making the conference a success. We hope that the friendly and stimulating atmosphere of this meeting will remain in the memory of all participants.

H. AMANN, G. P. GALDI, K. PILECKAS, V. A. SOLONNIKOV

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A PROBLEM OF EXPONENTIAL DECAY FOR NAVIER–STOKES EQUATIONS ARISING IN THE ANALYSIS OF RUGOSITY

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ABSTRACT

We are interested with the approximation of the flow past a plate covered by periodic asperities. The obtention of a good approximation reduces to two problems. The first problem, solved here, is a property of exponential decay of solutions of Navier–Stokes equations with additional Poiseuille terms in a semi-infinite channel which are periodic in the transverse directions. The second problem remains open.

1. INTRODUCTION

1.1. Flow past a plate covered with periodic asperities

We consider a viscous fluid occupying an infinite horizontal domain bounded by two plates, a plane one $\mathcal{P}$ and a rugose one $\mathcal{R}_\varepsilon$, covered with periodically distributed asperities of size ε, which is moving in a parallel direction to the first one with a constant velocity. In coordinates linked with the second one, the domain is

$$\mathcal{O}_\varepsilon = \{x \in \mathbb{R}^3 : x' \in \mathbb{R}^2, \eta_\varepsilon(x') < x_3 < \ell_3\},$$

where $x' = (x_1, x_2)$, $\ell_3 > 0$, $\varepsilon > 0$ and

$$\eta_\varepsilon(x') = \varepsilon\eta\Big(\frac{x'}{\varepsilon}\Big), \tag{1}$$

η being periodic with respect to x_1 and x_2 with periods ℓ_1 and ℓ_2. The velocity $u_\varepsilon = (u_{\varepsilon 1}, u_{\varepsilon 2}, u_{\varepsilon 3})$ and the pressure p_ε satisfy the stationary Navier–Stokes

* Corresponding author.

equations

$$-\nu\Delta u_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon + \nabla p_\varepsilon = 0, \qquad \nabla\cdot u_\varepsilon = 0,$$
$$u_\varepsilon|_{\mathcal{R}_\varepsilon} = 0, \qquad u_\varepsilon|_{\mathcal{P}} = g, \tag{2}$$

where $\nu > 0$ and $g = (g', 0)$ are constant, and they are assumed to be periodic with respect to x_1 and x_2 with periods $\varepsilon\ell_1$ and $\varepsilon\ell_2$. We assume $1/\varepsilon$ to be integer, which implies that η_ε, u_ε and p_ε are also periodic with periods ℓ_1 and ℓ_2.

We are looking for a non oscillating (that is independent of x'/ε) approximation of the velocity of the kind

$$u_\varepsilon(x) = \frac{x_3}{\ell_3}\, d_\varepsilon + g - d_\varepsilon + \mathcal{O}(\exp^{-c\,x_3/\varepsilon}), \tag{3}$$

outside a neighbourhood of $\mathcal{R}_\varepsilon$.

1.2. Exponential decay in a channel

Denoting $S = (0,\ell_1)\times(0,\ell_2)$, a semi-infinite channel is defined by

$$\Theta = \{x \in \mathbb{R}^3 : x' \in S,\ x_3 > 0\}$$

which is bounded on below by the part of plane

$$\Sigma = \{x \in \mathbb{R}^3 : x' \in S,\ x_3 = 0\}.$$

For Stokes equations, an approximation of type (3) of u_ε was deduced in (Amirat and Simon, 1996) from the following decay property: Any solution (Φ, Π) of Stokes equations in $\mathbb{R}^3_+$ which is periodic with respect to x_1 and x_2 and such that

$$\int_\Sigma \Phi_3\,\mathrm{d}x' = 0, \qquad \int_\Theta |\nabla\Phi|^2\,\mathrm{d}x < \infty \tag{4}$$

satisfies

$$|\nabla\Phi(x)| \leqslant c_\ell \|\Phi\|_{(L^2(\Sigma))^3} \exp(-c_\ell x_3), \quad \forall x_3 \geqslant 1. \tag{5}$$

The method to get these properties is recalled in Section 2.

For Navier–Stokes equations, as we will see in Section 3, the same method yields an auxiliary equation in the channel which is now

$$-\nu\Delta\phi + \phi\cdot\nabla\phi + \phi_3\frac{d}{\ell_3} + \left(\widehat{x_3}\frac{d}{\ell_3} + g - d\right)\cdot\nabla\phi + \nabla\pi = 0, \qquad \nabla\cdot\phi = 0,$$

where $d \in \mathbb{R}^3$, $d_3 = 0$ and $\widehat{x_3} = x_3$ if $x_3 \leqslant \ell_3$, $\widehat{x_3} = 0$ if $x_3 \geqslant 2\ell_3$. The first problem, which is solved in Section 4, is to get the estimate (5) under the hypothesis (4). A second problem, which remains open but which was solved in (Amirat and Simon, 1996) for Stokes equations, is to find d such that the limit of ϕ as x_3 goes to ∞ cancels.

Inequalities such as (5), which are called de Saint-Venant estimates, have been obtained for various problems. For Laplace equation, such an inequality was proved in the case of Dirichlet boundary conditions on the lateral boundary by J.-L. Lions (Lions, 1992, Theorem 10.1, p. 54) by using a lemma of L. Tartar; for periodic lateral boundary condition, see (Amirat and Simon, 1997, Lemma 4). For elasticity equations, the reader is referred for instance to (Oleinik *et al.*, 1992).

For Stokes equations, the case of Dirichlet lateral boundary condition is treated in (Galdi, 1994, Theorem 2.2, p. 319) (see also (Mikelić, 1995, Proposition 1, p. 1292) for a related problem); for the Neumann condition see (Lions, 1981); for periodic solutions see (Amirat and Simon, 1996, Lemma 4), which is slighly improved here. For Navier–Stokes equations with a Dirichlet lateral boundary condition see K. A. Ames and L. E. Payne (1989), G. P. Galdi (1994), C. O. Horgan and L. T. Wheeler (1978), O. A. Ladyzhenskaya and V. A. Solonnikov (1983); in the case of periodic solutions the reader is referred to (Amirat *et al.*, 1997).

2. THE CASE OF STOKES EQUATIONS

The domain $\mathcal{O}_\varepsilon$ is generated by periodic horizontal translations of the bounded domain

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3 : x' \in S,\ \eta_\varepsilon(x') < x_3 < \ell_3\}$$

which is bounded by the plates parts

$$R_\varepsilon = \{x : x' \in S,\ x_3 = \eta_\varepsilon(x')\}, \qquad P = \{x : x' \in S,\ x_3 = \ell_3\}$$

and by the (fictitious) lateral boundary $L_\varepsilon = \{x : x' \in \partial S,\ \eta_\varepsilon(x') < x_3 < \ell_3\}$. Let the space of periodic functions be defined by

$$H^m_{\text{per}}(\Omega_\varepsilon) = \{f \in H^m_{\text{loc}}(\mathcal{O}_\varepsilon) : f \in H^m(\Omega_\varepsilon),\\ f(x_1+\ell_1, x_2, x_3) = f(x_1, x_2+\ell_2, x_3) = f(x_1, x_2, x_3)\}$$

provided with the norm of $H^m(\Omega_\varepsilon)$. The profile of the asperities is assumed to be given by (1), with

$$\eta \in \text{Lip}_{\text{per}}(S), \qquad \varepsilon\|\eta\|_{L^\infty(S)} \leqslant \ell_3/2, \quad 1/\varepsilon \text{ is integer},$$

thus $|\eta(x') - \eta(y')| \leqslant c|x'-y'|$ and $\eta(x_1+\ell_1, x_2) = \eta(x_1, x_2+\ell_2) = \eta(x_1, x_2)$.

Let here $(u_\varepsilon, p_\varepsilon)$ be the unique solution of

$$\begin{cases} u_\varepsilon \in (H^1_{\text{per}}(\Omega_\varepsilon))^3, & p_\varepsilon \in L^2_{\text{per}}(\Omega_\varepsilon), \quad \displaystyle\int_{\Omega_\varepsilon} p_\varepsilon \,\mathrm{d}x = 0, \\ -\nu\Delta u_\varepsilon + \nabla p_\varepsilon = 0, & \nabla\cdot u_\varepsilon = 0, \quad u_\varepsilon|_{\mathcal{R}_\varepsilon} = 0, \quad u_\varepsilon|_{\mathcal{P}} = g. \end{cases}$$

Its approximation will involve a function (Φ, Π) in the "rugose half-space" $\widetilde{\mathcal{O}}_1$ which is bounded on below by the rugose plate $\mathcal{R} = \mathcal{R}_1$ with asperities of size $\varepsilon = 1$, that is

$$\widetilde{\mathcal{O}}_1 = \{x \in \mathbb{R}^3 : x' \in \mathbb{R}^2,\ x_3 > \eta(x')\}.$$

This domain is generated by periodic horizontal translations of the following semi-infinite channel with rugose bottom

$$\widetilde{\Omega}_1 = \{x \in \mathbb{R}^3 : x' \in S,\ x_3 > \eta(x')\}.$$

A unique pair (Ψ, Ξ) is defined in $\widetilde{\mathcal{O}}_1$, cf. (Amirat and Simon, 1996), by

$$\begin{cases} \Psi \in (H^1_{\text{per, loc}}(\widetilde{\Omega}_1))^3, & \nabla\Psi \in (L^2(\widetilde{\Omega}_1))^9, \quad \Xi \in L^2_{\text{per, loc}}(\widetilde{\Omega}_1), \\ -\nu\Delta\Psi + \nabla\Xi = 0, & \nabla\cdot\Psi = 0, \quad \Psi|_{R_1} = \eta g, \quad \displaystyle\int_{\widetilde{\Omega}_1} \Xi \,\mathrm{d}x = 0. \end{cases} \tag{6}$$

We will use its mean value on the cross section $\Sigma_{\overline{\eta}}$ where $\overline{\eta} = \max\{\eta(x') : x' \in S\}$, that is

$$b = \frac{1}{\ell_1 \ell_2} \int_S \Psi(x', \overline{\eta}) \,\mathrm{d}x'. \tag{7}$$

Remark. The condition $\int_{\widetilde{\Omega}_1} \Xi \,\mathrm{d}x = 0$ may be satisfied, although $\widetilde{\Omega}_1$ is not bounded, since the other properties in (6) imply $|\Xi(x) - \Xi_\infty| \leqslant c' \exp(-cx_3)$.

Let us check that $b_3 = 0$. The incompressibility gives

$$0 = \int_{\Omega_1} \nabla\cdot\Psi = \int_{P\cup R_1\cup L_1} \Psi\cdot n \,\mathrm{d}s,$$

where n is the unit outward vector field on the boundary of Ω_1. By the periodicity, the integral over the lateral boundary L_1 vanishes. On R_1, $\Psi = \eta g$ by the boundary condition and $n\,\mathrm{d}s = (\partial_1\eta, \partial_2\eta, -1)\,\mathrm{d}x'$, thus $\Psi\cdot n\,\mathrm{d}s = \frac{1}{2}\nabla\cdot(g\eta^2)$ and its integral cancels. Finally, the integral over P gives

$$\int_S \Psi_3(x', \ell_3) \,\mathrm{d}x' = 0. \tag{8}$$

In this calculus, ℓ_3 may be replaced by $\overline{\eta}$, which gives $b_3 = 0$.

Since $g = (g', 0)$, $b = (b', 0)$ and the map $g' \mapsto b'$ is linear continuous from $\mathbb{R}^2$ into itself, there exists a linear continuous map B from $\mathbb{R}^3$ into itself such that, denoting $\{e_1, e_2, e_3\}$ the basis of $\mathbb{R}^3$,

$$B\,g = b, \qquad B\,e_3 = 0.$$

Remark. The matrix of B is made up of the column vectors b^1, b^2 and 0, where b^1 and b^2 are defined by (6) and (7), with respectively $g = e_1$ and $g = e_2$.

Remark. The function Ψ is independent of ν (only Ξ depends on ν: indeed, the equation in (6) may be written as $-\Delta\Psi + \nabla(\Xi/\nu) = 0$). In (7), the choice of $\overline{\eta}$ is arbitrary since $\int_S \Psi_3(x', x_3)\,\mathrm{d}x'$ does not depend on $x_3 \geqslant \overline{\eta}$.

Now, we are able to give an approximation of the velocity and of the pressure up to an error exponentially decreasing with ε.

PROPOSITION 1. *Let* $d_\varepsilon = (I - \frac{\varepsilon}{\ell_3} B)^{-1} g$. *For all* $\varepsilon(\overline{\eta} + 1) \leqslant x_3 \leqslant \ell_3$ *and for all* $\alpha \geqslant 0$,

$$u_\varepsilon(x) = \frac{x_3}{\ell_3}\, d_\varepsilon + g - d_\varepsilon + w_\varepsilon(x), \tag{9}$$

$$|\partial^\alpha w_\varepsilon(x)| + |\partial^\alpha p_\varepsilon(x)| \leqslant \varepsilon |g|\, c_{\alpha,\ell,\eta} \exp\Big(-\frac{c_\ell\, x_3}{\varepsilon}\Big). \tag{10}$$

We denote $c_\ell, c_{\ell,\eta}, \ldots$ real numbers which are independent of the other data, but which may change from an inequality to the other one.

Remark. Let us notice that $d_\varepsilon = g + \frac{\varepsilon}{\ell_3}(I - \frac{\varepsilon}{\ell_3} B)^{-1} B g$.

We will recall now the main lines of the proof, which is given in (Amirat and Simon, 1996), in order to see how we could expect to extend it to Navier–Stokes equations.

Outlines of the proof of Proposition 1. Construction of the corrector and of the residue. We define a corrector $(\psi_\varepsilon, \xi_\varepsilon)$ in the "rugose half-space" $\widetilde{\mathcal{O}}_\varepsilon$ by

$$\psi_\varepsilon(x) = \frac{\varepsilon}{\ell_3}\Big(B d_\varepsilon - \Psi_{d_\varepsilon}\Big(\frac{x}{\varepsilon}\Big)\Big), \qquad \xi_\varepsilon(x) = -\frac{1}{\ell_3}\Xi_{d_\varepsilon}\Big(\frac{x}{\varepsilon}\Big), \tag{11}$$

where $(\Psi_{d_\varepsilon}, \Xi_{d_\varepsilon})$ denotes the solution of (6) for $g = d_\varepsilon$. It satisfies

$$-\nu\Delta\psi_\varepsilon + \nabla\xi_\varepsilon = 0, \qquad \nabla\cdot\psi_\varepsilon = 0,$$
$$\psi_\varepsilon|_{\mathcal{R}_\varepsilon} = \frac{\varepsilon}{\ell_3} B d_\varepsilon - \frac{\eta_\varepsilon}{\ell_3} d_\varepsilon, \qquad \int_S \psi_\varepsilon(x', \varepsilon\overline{\eta})\,\mathrm{d}x' = 0.$$

Then, w_ε being given by (9), we define a residue $(\chi_\varepsilon, \sigma_\varepsilon)$ in $\mathcal{O}_\varepsilon$ by $w_\varepsilon = \psi_\varepsilon + \chi_\varepsilon$, $p_\varepsilon = \xi_\varepsilon + \sigma_\varepsilon$. It satisfies

$$-\nu\Delta\chi_\varepsilon + \nabla\sigma_\varepsilon = 0, \qquad \nabla\cdot\chi_\varepsilon = 0, \qquad \chi_\varepsilon|_{\mathcal{P}} = -\psi_\varepsilon|_{\mathcal{P}}, \qquad \chi_\varepsilon|_{\mathcal{R}_\varepsilon} = 0 \quad (12)$$

since $w_\varepsilon|_{\mathcal{P}} = 0$ and $w_\varepsilon|_{\mathcal{R}_\varepsilon} = d_\varepsilon - g - \frac{\eta_\varepsilon}{\ell_3} d_\varepsilon$ (thus $\chi_\varepsilon|_{\mathcal{R}_\varepsilon} = d_\varepsilon - \frac{\varepsilon}{\ell_3} B d_\varepsilon - g$).

Exponential decay. To get a function defined in the half-space $\mathbb{R}^3_+$, we consider

$$\Phi(x) = Bd_\varepsilon - \Psi_{d_\varepsilon}(x', x_3 + \bar{\eta}), \qquad \Pi(x) = -\Xi_{d_\varepsilon}(x', x_3 + \bar{\eta}).$$

By definition (6) of (Ψ, Ξ) and by definition of B, it satisfies

$$-\nu\Delta\Phi + \nabla\Pi = 0, \qquad \nabla\cdot\Phi = 0, \qquad \int_S \Phi(x', 0)\,\mathrm{d}x' = 0.$$

By Lemma 4 of (Amirat and Simon, 1996), this implies, for all $x_3 \geqslant 1$,

$$|\partial^\alpha\Phi(x)| + |\partial^\alpha\Pi(x)| \leqslant c_{\alpha,\ell}\|\Phi_0\|_{(L^2(S))^3}\exp(-c_\ell\, x_3),$$

where $\Phi_0(x') = \Phi(x', 0)$. Since $\|\Phi_0\|_{(L^2(S))^3} \leqslant c_{\ell,\eta}|d_\varepsilon|$, (11) gives, for all $x_3 \geqslant \varepsilon(\bar{\eta} + 1)$,

$$|\partial^\alpha\psi_\varepsilon(x)| + |\partial^\alpha\xi_\varepsilon(x)| \leqslant c_{\alpha,\ell,\eta}\varepsilon|d_\varepsilon|\exp\Big(-\frac{c_\ell x_3}{\varepsilon}\Big).$$

Then, $|\psi_\varepsilon|_P| \leqslant c_{\alpha,\ell,\eta}\varepsilon|d_\varepsilon|\exp(-c_\ell/\varepsilon)$ and (12) gives, in the whole set $\mathcal{O}_\varepsilon$,

$$|\partial^\alpha\chi_\varepsilon(x)| + |\partial^\alpha\sigma_\varepsilon(x)| \leqslant c_{\alpha,\ell,\eta}\varepsilon|d_\varepsilon|\exp\Big(-\frac{c_\ell}{\varepsilon}\Big).$$

This, together with the previous inequality, proves (10). □

3. THE PROBLEMS FOR NAVIER–STOKES EQUATIONS

3.1. *Approximation of the velocity*

Let now u_ε satisfy the Navier–Stokes equations and let us again look for an expansion of the type (9), for a convenient d_ε, that is

$$u_\varepsilon(x) = \frac{x_3}{\ell_3} d_\varepsilon + g - d_\varepsilon + w_\varepsilon(x),$$

where $d_{\varepsilon 3} = 0$. Using this in the equation (2), we get

$$\begin{cases} -\nu\Delta w_\varepsilon + w_\varepsilon\cdot\nabla w_\varepsilon + w_{\varepsilon 3}\dfrac{d_\varepsilon}{\ell_3} + \Big(\dfrac{x_3}{\ell_3}d_\varepsilon + g - d_\varepsilon\Big)\cdot\nabla w_\varepsilon + \nabla p_\varepsilon = 0, \\ \nabla\cdot w_\varepsilon = 0, \qquad w_\varepsilon|_{\mathcal{P}} = 0, \qquad w_\varepsilon|_{\mathcal{R}_\varepsilon} = -\dfrac{\eta_\varepsilon}{\ell_3}d_\varepsilon - g + d_\varepsilon. \end{cases} \quad (13)$$

The problem is to find d_ε, depending on g and ν, such that the solution $(w_\varepsilon, p_\varepsilon)$ of (13) satisfies the decay property (10).

Instead of using a corrector (Ψ, Ξ) in the rugose half-space $\widetilde{\mathcal{O}}_1$ with asperities of size 1 as for Stokes equations, here we consider directly a corrector $(\psi_\varepsilon, \xi_\varepsilon)$ in the rugose half-space $\widetilde{\mathcal{O}}_\varepsilon$ with asperities of size ε. It is defined by

$$\begin{cases} \psi_\varepsilon \in (H^1_{\text{per, loc}}(\widetilde{\Omega}_\varepsilon))^3, \qquad \nabla\psi_\varepsilon \in (L^2(\widetilde{\Omega}_\varepsilon))^9, \qquad \xi_\varepsilon \in L^2_{\text{per, loc}}(\widetilde{\Omega}_\varepsilon), \\ -\nu\Delta\psi_\varepsilon + \psi_\varepsilon\cdot\nabla\psi_\varepsilon + \psi_{\varepsilon 3}\dfrac{d_\varepsilon}{\ell_3} + \Big(\dfrac{\widehat{x}_3}{\ell_3}d_\varepsilon + g - d_\varepsilon\Big)\cdot\nabla\psi_\varepsilon + \nabla\xi_\varepsilon = 0, \\ \nabla\cdot\psi_\varepsilon = 0, \qquad \psi_\varepsilon|_{\mathcal{R}_\varepsilon} = -\dfrac{\eta_\varepsilon}{\ell_3}d_\varepsilon - g + d_\varepsilon, \end{cases} \tag{14}$$

where $\widehat{x}_3 = x_3$ if $x_3 \leqslant \ell_3$, $\widehat{x}_3 = 2\ell_3 - x_3$ if $\ell_3 \leqslant x_3 \leqslant 2\ell_3$ and $\widehat{x}_3 = 0$ else. The existence of such a corrector is obtained, provided that $\varepsilon^2 d_\varepsilon$ is small enough (i.e. $\varepsilon^2 d_\varepsilon \leqslant c_{\nu,\ell}$), by passing to the limit as $m \to \infty$ on the solution defined in the horizontal strip $\widetilde{\mathcal{O}}_\varepsilon \cap \{x : x_3 \leqslant m\}$ which cancels for $x_3 = m$.

Then, a residue $(\chi_\varepsilon, \sigma_\varepsilon)$ is defined in $\mathcal{O}_\varepsilon$ by

$$w_\varepsilon = \psi_\varepsilon + \chi_\varepsilon, \qquad p_\varepsilon = \xi_\varepsilon + \sigma_\varepsilon.$$

It satisfies

$$\begin{cases} -\nu\Delta\chi_\varepsilon + \chi_\varepsilon\cdot\nabla\chi_\varepsilon + \psi_\varepsilon\cdot\nabla\chi_\varepsilon + \chi_\varepsilon\cdot\nabla\psi_\varepsilon + \chi_{\varepsilon 3}\dfrac{d_\varepsilon}{\ell_3} \\ \quad + \Big(\dfrac{x_3}{\ell_3}d_\varepsilon + g - d_\varepsilon\Big)\cdot\nabla\chi_\varepsilon + \nabla\sigma_\varepsilon = 0, \\ \nabla\cdot\chi_\varepsilon = 0, \qquad \chi_\varepsilon|_{\mathcal{P}} = -\psi_\varepsilon|_{\mathcal{P}}, \qquad \chi_\varepsilon|_{\mathcal{R}_\varepsilon} = 0. \end{cases}$$

Should $\psi_\varepsilon(x)$ decay exponentially fast with respect to x_3/ε, then $|\psi_\varepsilon|_{\mathcal{P}}| \leqslant c'\exp(-c/\varepsilon)$ which would imply $|\chi_\varepsilon| \leqslant c''\exp(-c/\varepsilon)$ in the whole domain $\mathcal{O}_\varepsilon$. Therefore, the remainder $(w_\varepsilon, p_\varepsilon)$ would decay exponentially fast in the sense of (10), which is our final goal.

To get this exponential decay of $\psi_\varepsilon(x)$, the *first problem* is to prove that

$$|\nabla\psi_\varepsilon(x)| \leqslant c'\exp\Big(-\frac{cx_3}{\varepsilon}\Big). \tag{15}$$

This property implies the existence of $\psi_{\varepsilon,\infty} \in \mathbb{R}^3$ such that $|\psi_\varepsilon(x) - \psi_{\varepsilon,\infty}| \leqslant c'\exp(-c/\varepsilon)$. The *second problem*, which remains open, is to find d_ε, depending on g and ν, such that $\psi_{\varepsilon,\infty} = 0$ or at least $|\psi_{\varepsilon,\infty}| \leqslant c'\exp(-c/\varepsilon)$. This is equivalent to find d_ε such that the solution of (14) satisfies

$$\int_S \psi_\varepsilon(x', \ell_3)\,\mathrm{d}x' = 0$$

or the weaker condition $\int_S \psi_\varepsilon(x', \ell_3)\,\mathrm{d}x' \leqslant c'\exp(-c/\varepsilon)$.

3.2. Solution of the first problem

The exponential decay (15) follows from the decay property proved in the next section, and more precisely from Theorem 2 applied to

$$(\phi, \pi)(x) = (\psi_\varepsilon, \xi_\varepsilon)(x', x_3 + \varepsilon\bar{\eta})$$

and to $L = \max(\varepsilon\ell_1, \varepsilon\ell_2)$.

The assumption $\int_\Sigma \phi_3 = 0$, that is $\int_S \psi_{\varepsilon 3}(x', \varepsilon\bar{\eta})\,\mathrm{d}x' = 0$ follows from the condition $\psi_\varepsilon|_{\mathcal{R}_\varepsilon} = \eta_\varepsilon\, d_\varepsilon/\ell_3 + g - d_\varepsilon$ (the proof is similar to the one of (8)).

4. A RESULT OF EXPONENTIAL DECAY

Let (ϕ, π) satisfy

$$\begin{cases} \phi \in (H^1_{\text{per, loc}}(\mathbb{R}^3_+))^3, \qquad \pi \in L^2_{\text{per, loc}}(\mathbb{R}^3_+), \qquad \displaystyle\int_\Theta |\nabla\phi|^2 \leqslant E, \\ -\nu\Delta\phi + \phi\cdot\nabla\phi + \phi_3 b + a\cdot\nabla\phi + \nabla\pi = 0, \qquad \nabla\cdot\phi = 0, \\ \displaystyle\int_\Sigma \phi_3\,\mathrm{d}x' = 0, \qquad |\bar{\phi}(0)| \leqslant E, \end{cases} \tag{16}$$

where

$$b \in \mathbb{R}^3, \qquad b_3 = 0,$$

$$a = a(x_3), \qquad a_3 = 0, \qquad a = 0 \ \text{ in } [2\ell_3, \infty), \qquad a \in (W^{1,\infty}((0,\infty)))^3,$$

and let $L = \max(\ell_1, \ell_2)$, where ℓ_1 and ℓ_2 are the periods with respect to x_1 and x_2. Then, the following decay property holds:

THEOREM 2. *Assume* $L \leqslant \frac{1}{2\kappa}\sqrt{\frac{\nu}{|b|}}$, *where* κ *is given by* (18). *Then, for all* $x_3 \geqslant L$,

$$|\nabla\phi(x)| \leqslant \mu\exp\Big(-\frac{\lambda x_3}{L}\Big) \tag{17}$$

with λ *and* μ *independent of* L *and* ϕ *(they may depend on* E, ν, a *and* b*).*

A similar result was proved in (Amirat *et al.*, 1997) for Navier–Stokes equations, that is without the terms $\phi_3 b + a\cdot\nabla\phi$. We will give the outlines of the proof with these additional terms, which relies on similar method than this of (Amirat *et al.*, 1997), to which the reader is referred for details.

At first, we will prove that the mean value $\bar{\phi}(x_3)$ of ϕ over a cross section possesses a limit at infinity. Then, we will use this property to adapt the method used (cf. (Ames and Payne, 1989; Galdi, 1994; Horgan and Wheeler, 1978;

Ladyzhenskaya and Solonnikov, 1983)) for Dirichlet condition to the periodic one, by using the Poincaré–Wirtinger inequality instead of the Poincaré one.

We denote for $t \geqslant 0$,

$$\Theta_t = \{x \in \mathbb{R}^3 : x' \in S,\ x_3 > t\},$$

$$\Sigma_t = \{x \in \mathbb{R}^3 : x' \in S,\ x_3 = t\},$$

(therefore $\Theta = \Theta_0$, $\Sigma = \Sigma_0$), $\partial_i = \partial/\partial_{x_i}$ and $\nabla_{x'} = (\partial_1, \partial_2)$. The mean value being defined by

$$\bar{\phi}(t) = \frac{1}{\ell_1 \ell_2} \int\limits_S \phi(x', t)\,\mathrm{d}x',$$

the Poincaré–Wirtinger inequality on Σ_t gives

$$\|\phi - \bar{\phi}\|_{(L^2(S))^3} \leqslant \kappa L \|\nabla_{x'}\phi\|_{(L^2(S))^6}, \tag{18}$$

where κ is a universal constant. In fact, one could choose here κ equal to the best constant for the Poincaré–Wirtinger inequality in the disk of surface 1, and $L = \sqrt{\ell_1 \ell_2}$. This follows from the fact that the constant for any set S of given surface can be bounded by the constant for a disk of same surface. An easier method consists to use a similarity with respect to x_1 with ratio ℓ_2/ℓ_1 and to use an inequality in a square.

Let us now prove the following result.

LEMMA 3. *For all* $x_3 > 0$,

$$\bar{\phi}_3(x_3) = 0 \tag{19}$$

and there exists $\bar{\phi}_\infty \in \mathbb{R}^3$ *such that*

$$|\bar{\phi}(x_3) - \bar{\phi}_\infty| \leqslant \frac{\kappa^2 L^2}{\nu \ell_1 \ell_2} \int\limits_{\Theta_{x_3}} |\nabla\phi|^2. \tag{20}$$

Let us remark that the right-hand side in (20) goes to 0 as $x_3 \to \infty$.

Proof. Proof of (19). Since ϕ is periodic with respect to x_1 and x_2, for all positive t we have $\int_S \partial_1\phi(x', t)\,\mathrm{d}x' = \int_S \partial_2\phi(x', t)\,\mathrm{d}x' = 0$. Therefore, $\nabla\cdot\phi = 0$ implies

$$0 = \int\limits_S \nabla \cdot \phi\,(x', t)\,\mathrm{d}x' = \int\limits_S \partial_3\phi_3\,(x', t)\,\mathrm{d}x'$$