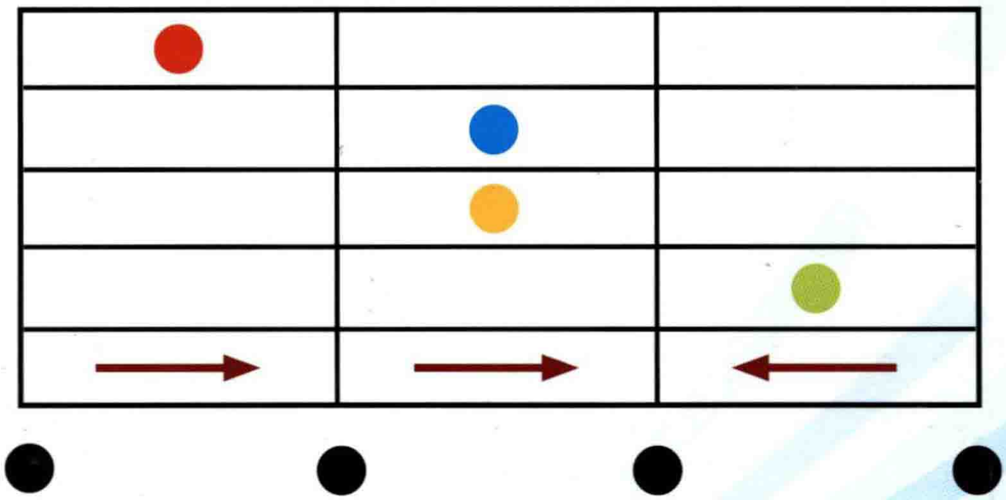
THE (1+1)-NONLINEAR UNIVERSE OF THE PARABOLIC MAP AND COMBINATORICS



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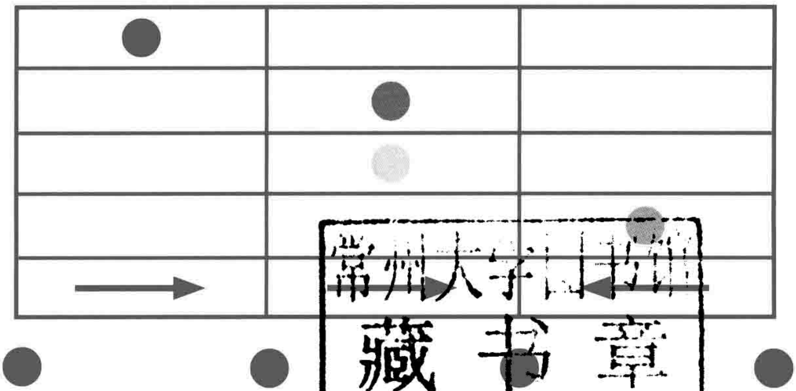

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NEW JERSEY · LONDON · SINGAPORE · BEIJING · SHANGHAI · HONG KONG · TAIPEI · CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Louck, James D.

The (1+1)-nonlinear universe of the parabolic map and combinatorics / by James D. Louck (Retired Los Alamos National Laboratory Fellow, USA), Myron L. Stein.

pages cm

Includes bibliographical references and index.

ISBN 978-9814632416 (hardcover : alk. paper)

1. Nonlinear theories. 2. Chaotic behavior in systems. 3. Combinatorial analysis. 4. Mathematical analysis. I. Stein, M. L. II. Title. III. Title: Nonlinear universe of the parabolic map and combinatorics.

QA427.L68 2015

511'.6--dc23

2014042607

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Printed in Singapore by Mainland Press Pte Ltd.

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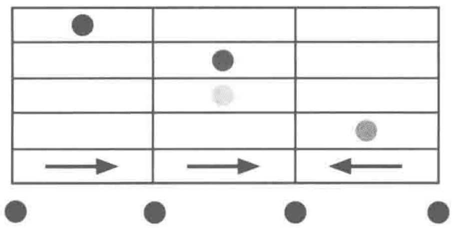


To the memory of Robert L. Bivins, Nicholas C. Metropolis, and Myron L. Stein, without whom this monograph could not have been completed.

Preface

The original motivation for this monograph was to set forth the early contributions from the Theoretical Division at Los Alamos National Laboratory to the foundations of chaos theory. Overviews of work done up to 1983 have already been given in LA–2305,1959 and in LA–9705,1983, which are available electronically on request from the Laboratory. These reports remark on the foundations of the subject as set forth in early papers by Stein and Ulam [1], N. Metropolis *et al* [2-3], Feigenbaum [4-9, 12], Feigenbaum *et al* [10], Beyer and Stein [11], Beyer *et al* [13], Stein [14], and the book by Bivens *et al* [15]. These are the primary references leading to the viewpoints developed in this monograph. The evolution of ideas beginning with the above references is an important ingredient of this monograph. It is this aspect that is focused on in the Preface, but this is intertwined by a preview of a major shift in viewpoint that developed as the writing progressed.

Principal properties promoted and developed by Bivins *et al* [16] are those of the inverse graph, which for a general function f with real values $f(x)$ is a collection of single-valued complex functions called branches. For the case at hand, the basic function is the parabola p_ζ, which is defined by its set of values $p_\zeta(x) = \zeta\, x(2-x)$, $x \in (-\infty, \infty)$. The parameter ζ is, for the most part, taken to be real with values of ζ in the closed interval $\zeta \in [0,2]$. (It turns out, however, that all real values $\zeta \in (-\infty,\infty)$ are important.) This method based on properties of the inverse graph was itself motivated by the discovery that the inverse graph had the property of being sometimes complex and sometimes real, but with the extraordinary property that each such inverse function becomes real at a characteristic value of $\zeta \in [0,2]$, and remains real for all greater values of ζ. Thus, a theory emerged that was based on function composition, one that also allowed the creation of objects such as curves and fixed points.

The major shift in viewpoint occurred when an algorithm was discovered during the write-up of the monograph that allowed the generation of the inverse graph for $n-1$ to n. This placed the subject clearly in the arena of a complex adaptive system, where a complex adaptive system is taken to be a system whereby a few principal axioms lead to a system rich in structure and predictive power. For the problem at hand, this was realized by some simple implementable rules, ones that could also be calculated numerically and verified visually. Thus, the idea of an algorithmic-computer-generated inverse graph had evolved that fits well with the notion of a complex adaptive system. But what about applications and predictability?

The complex adaptive system viewpoint is further enriched by properties of the inverse graph that can be interpreted in terms of combinatorial concepts such as a total order relation on all branches of the inverse graph that exist at a given value of ζ, an order relation that is never violated, up to and including all positive values of ζ. Moreover, this labeling of branches of the inverse group can be realized by hook tableaux, which are special Young standard tableaux, or, equivalently, by special Gelfand-Tsetlin patterns. Such patterns can be realized as isotropic quantum oscillators.

The complex system applications do not end here; they continue still into biology and beyond: see Bell *et al* [20] and Bell and Torney [21] with yet further applications to Galois groups by Byers and Louck [23] and to Conway numbers by Byers and Louck [24-25].

Most importantly for this monograph the issue of an application to General Relativity arises based on the mathematical operation of function composition; the case for a complex adaptive system has been established. Whether or not it provides any meaningful insights into General Relativity remains to be seen. The authors have no experience working in General Relativity other than a general introduction, which is inadequate for such judgments. But there is still an obligation to point out the possibilities. It appears that the existence of **A Fully Deterministic Chaos Theory** is a **basic property** with a potential application to General Relativity.

The first author takes full responsibility for the viewpoints presented in this monograph. It is, of course, the case that these viewpoints could not have emerged without the extraordinary interaction between computer calculations and the development of theory.

A somewhat unusual style of presentation has been utilized in this monograph. Many pictures of inverse graphs at various parameter-values $\zeta_1 < \zeta_2 < \cdots < \zeta_t < \cdots$ are given that illustrate crucial properties of the ζ-parameter evolution of the inverse graph. Thus, the notion that the system under study is a complex adaptive system is re-enforced by computer calculations in which the inverse graph exhibits the predicted properties. Sufficiently many computer graphs are included, as needed to exhibit a particular property. For a vivid mental picture, it is often useful to think of ζ as time. It is in this time-evolution of the n-th iterate of the inverse graph that the classification by words on two letters comes into play, their fundamental role being to enumerate the branches of the inverse graph. The patterns exhibited by explicit computer computations of the shape of graphs and the expression of their explicit mathematical forms is a nice example of how one mode of presentation generates and re-enforces insights into the other. This accounts for the dedication of this work to the memories of R. L. Bivins, Nicholas C. Metropolis, and Myron L. Stein. World Scientific graciously allowed the inclusion of Myron's name on the cover, since his computational contribution was completed before his death. It is quite impossible to express the compassion and support of Editor Lai Fun Kwong.

The organization of this work, the many pictures of the inverse graph aside, is quite standard, as detailed in the **Contents.** It is emphasized that this monograph is far too technical and detailed to be a textbook. It is intended for readers with a perchance for the unusual and unexpected. Most will probably have a background in physics or mathematics.

This work could not have been completed without 54 years of enduring patience and endearing love of my wife Marge and the expert computer maintenance support of our son Tom. Thanks are given to David C. Torney, Peter W. Milonni, and Michael M. Nieto (deceased 2013) for many useful discussions on the foundations of mathematics and physics. Also, thanks to Librarians Michelle Mittrach and Kathy Varjabedian who diligently provided electronic copies of references. The viewpoints and attributions expressed herein are mine alone.

James D. Louck

Contents

Chapter 1

Introduction and Point of View

In this opening chapter, a synthesis is given of the results found in Refs. [1-5, 15-19]. The ideas, procedures, and definitions introduced in this Chapter are drawn from these references. Slight variations in notations may occur. The idea of this overview is to capture many of the over-riding features of the so-called ζ-evolution of the various graphs without giving all the many details needed for their complete description.

1.1 Function Composition and Graphs

The principal mathematical operation that produces most curves generated and discussed in this monograph is the operation on pairs of functions known as *composition*. The composition of a pair of functions f and g is denoted by $f \circ g$. It is defined by giving its value, denoted $(f \circ g)(x)$, in terms of the values of the functions f and g, as expressed by

$$(f \circ g)(x) = f\Big(g(x)\Big). \tag{1.1}$$

Thus, $(f \circ g)(x)$ is the value of $f(x)$ at $x = g(x)$. The operation of composition is noncommutative, but associative:

$$f \circ g \neq g \circ f; \quad (f \circ g) \circ h = f \circ (g \circ h), \tag{1.2}$$

as verified directly from the definition (1.1).

The composition of pairs of functions generalizes directly to that of the composition of arbitrarily many functions:

$$\begin{aligned}(f_1 \circ f_2 \circ f_3)(x) &= f_1\Bigg(f_2\Big(f_3(x)\Big)\Bigg), \\ (f_1 \circ f_2 \circ f_3 \circ f_4)(x) &= f_1\Bigg(f_2\bigg(f_3\Big(f_4(x)\Big)\bigg)\Bigg),\end{aligned} \tag{1.3}$$

$$\vdots$$

$$(f_1 \circ \cdots \circ f_{n-2} \circ f_{n-1} \circ f_n)(x) = f_1\Big(\cdots f_{n-2}\Big(f_{n-1}\Big(f_n(x) \Big)\Big) \cdots \Big).$$

Because the rule of composition is associative, no additional parenthesis pairs are needed in the left-hand side of these relations. There are n parenthesis pairs ( ) on the right-hand side: n left parentheses (one following each f_i, and each matched with a right parenthesis), thus constituting a parenthesis pair ( ), where all n right parentheses occur in succession at the right-most end of each of relations (1.3).

The inverse of a function f with values $f(x)$ is denoted by f^{-1} and is defined here to be a single-valued function with values denoted by $f^{-1}(x)$ such that

$$f\Big(f^{-1}(x)\Big) = f^{-1}\Big(f((x)\Big) = x. \tag{1.4}$$

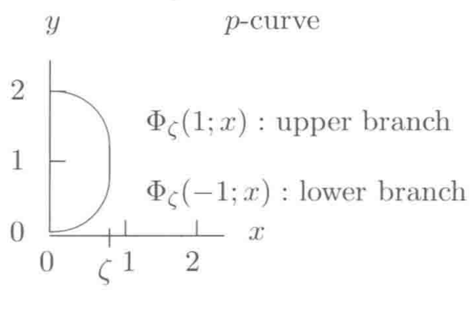
Thus, the inverses to f are solutions of the equation $f(y(x)) = x$, and in general there can be several distinct solutions; careful attention must be paid to the domains of definition of f and f^{-1}. In this monograph, distinct inverses to a given single real-valued function f are called **branches.** An inverse f^{-1} to f can also be defined by the composition rule $f^{-1} \circ f = f \circ f^{-1} = I$, where I is the identity function with values $I(x) = x$. The interest here is not with all the subtleties that arise in considering collections of functions and their compositions, but, rather, with the properties of the n-fold composition of a single function — the parabola defined by

$$p_\zeta(x) = \zeta x(2 - x),\ \zeta \in (0, \infty);\ x \in (-\infty, \infty). \tag{1.5}$$

Most of the interest of the present monograph is directed toward the development of the properties of the 2^n-fold compositions of the two branches of the inverse function to $p_\zeta(x)$ as defined by

$$\begin{aligned} \Phi_\zeta(1;x) &= 1 + \sqrt{1 - \frac{x}{\zeta}};\ \ \Phi_\zeta(-1;x) = 1 - \sqrt{1 - \frac{x}{\zeta}}, \\ &\zeta \in (0, \infty),\ x \in (-\infty, \zeta). \end{aligned} \tag{1.6}$$

Each of these branches is, of course, a real single-valued function of x in the domain $x \in (-\infty, \zeta)$, and the two functions join smoothly at $x = \zeta$ to constitute what will be called a p-curve. A p-curve is the joining of two branches as illustrated in the following schematic picture for the (x, y)-planar graph of the branches $\Phi_\zeta(1;x)$ and $\Phi_\zeta(-1;x)$ for $x \in (0, \zeta]$:



(1.7)

This picture depicts a **right-moving** p-curve with increasing ζ. The general polynomials of interest are the real polynomials of degree 2^n in x defined by the n-fold composition of p_ζ:

$$p_\zeta^n(x) = \Big(p_\zeta \circ p_\zeta \circ \cdots \circ p_\zeta\Big)(x) = p_\zeta\bigg(\cdots\Big(p_\zeta\big(p_\zeta(x)\big)\Big)\cdots\bigg), \tag{1.8}$$

where there are n parenthesis pairs in this expression for an n-fold composition of one and the same parabola function p_ζ. It is very important to observe that the parameter ζ is fixed at the same value in the composition (1.8). Thus, while it is allowed that ζ be any value $\zeta \in (0,\infty)$, the operation of composition is to be effected only for specified ζ in its domain of definition, as illustrated by

$$\begin{aligned} p_\zeta^2(x) &= \Big(p_\zeta \circ p_\zeta\Big)(x) = \zeta\, x(2-x)\Big|_{x=\zeta\, x(2-x)} \\ &= \zeta^2\, x(2-x)\Big(2-\zeta\, x(2-x)\Big). \end{aligned} \tag{1.9}$$

A very useful rule satisfied by such compositions is:

$$\begin{aligned} p_\zeta^n(x) &= \Big(p_\zeta^{n-m} \circ p_\zeta^m\Big)(x) = p_\zeta^{n-m}\Big(p_\zeta^m(x)\Big), \\ &\qquad m = 1, 2, \ldots, n-1, \\ p_\zeta^1(x) &= p_\zeta(x) = \zeta\, x(2-x). \end{aligned} \tag{1.10}$$

The n-fold iterate $p_\zeta^n(x)$ of $p_\zeta^1(x)$ is a polynomial of degree 2^n in the variable x and degree $2^n - 1$ in the parameter ζ. Thus, the polynomial is of the form

$$p_\zeta^n(x) = \sum_{k=0}^{2^n} a_k^{(n)}(\zeta)\, x^{2^n-k}, \tag{1.11}$$

where the coefficients are real polynomials in the parameter ζ with leading coefficient $a_0^{(n)}(\zeta) = 2^n - 1$ and successive coefficients of lower degree. A recurrence relation for the polynomials is given by

$$p_\zeta^n(x) = (p_\zeta^{n-1} \circ p_\zeta^1)(x) = p_\zeta^{n-1}\Big(p_\zeta^1(x)\Big). \tag{1.12}$$

Thus, an explicit recurrence for the coefficients $a_k^{(n)}(\zeta)$ themselves can be obtained, if desired, by combining relation (1.12) and (1.11) with the appropriate relations from (1.10). The main point is: *The polynomials $p_\zeta^n(x)$ are uniquely defined for all positive n.*

The graph H_ζ^n of interest is defined as the set of points in the Cartesian plane $\mathbb{R}^2$ given by

$$H_\zeta^n = \Big\{\Big(x,\, p_\zeta^n(x)\Big)\Big|\, x \in [0,\,\infty)\Big\},\ \zeta \in (0,\infty). \tag{1.13}$$

Many of the interesting features of this graph make their appearance for $x \in [0, 2]$, although other domains, even including negative x, are of interest. A principal feature of all graphs presented in Chapters 5-7 is that they are presented at a value of the parameter ζ that is **specified** (fixed). The values of x then determine the basic **shape** of the underlying curve in the (x, y)-plane for the specified value of ζ; this set of real points constitute the graph H_ζ^n: It is a continuous smooth curve (all derivatives exist at all points) in $\mathbb{R}^2$. This set of points is also called the *shape of the graph* H_ζ^n at ζ.

As the parameter ζ changes continuously, the shape of the curve, which is denoted by $\mathcal{H}_\zeta^n$, changes smoothly. In particular, the change in shape for increasing ζ is called the *ζ-evolution* of the curve (or graph). Indeed, it is often very useful to think of ζ as a time-like parameter; hence, the shape $\mathcal{H}_\zeta^n$ is a "snapshot" of the graph at a given time, and the ζ-evolution is the nonlinear time progression of the graph. The ζ-evolution of the graph is unexpectedly elegant, expressing its unfolding shape in terms of the creation of new "subcurves" and their symmetry. It is the purpose of this monograph to give its description for all n.

There is a simple underlying reason for the origin of the features appearing in the ζ-evolution of the graph H_ζ^n: *This is revealed in the structure of the inverse graph.* If $H_f = \{(x, f(x)) \,|\, x \in D_f\}$ is the graph of a real single-valued function f with values $f(x)$ and domain of definition $x \in D_f \subseteq \mathbb{R}$, then, by definition, the set of points $H_{f^{-1}} = \{(x, f^{-1}(x)) \,|\, x \in D_{f^{-1}}\}$, where $D_{f^{-1}} \subseteq \mathbb{R}$ is the domain of definition of the branch f^{-1}, constitutes a subgraph of the inverse graph. But there is such an inverse graph for each distinct inverse function f^{-1} of f; hence, it is the union $\cup_{f^{-1}} H_{f^{-1}}$ over all distinct inverse subgraphs that constitutes the full inverse graph to H_f. This simple description of the inverse graph holds unambiguously for the inverse graph of the n-fold composition of the parabolic map $p_\zeta(x) = \zeta\, x(2 - x)$, although care must be taken in defining the inverse function. In terms of these notations, the graph H_ζ^n is given by

$$H_\zeta^n = H_{p_\zeta^n} = \left\{ (x,\, p_\zeta^n(x)) \,\middle|\, x \in [0,\, \infty) \right\}, \tag{1.14}$$

where the n-fold composition of the basic parabola $p_\zeta^1(x) = p_\zeta(x) = \zeta\, x(2-x)$ is defined in (1.8). It is the inverse graph to $H_{p_\zeta^n}$ that is sought for each specified $\zeta \in (0, \infty)$. In terms of the present notations, the inverse graph is denoted by $H_{f_\zeta^{-1}}$, where $f = p_\zeta^n$. For the case at hand, this somewhat awkward notation is replaced by

$$G_\zeta^n = H_{f_\zeta^{-1}} \Big|_{f = p_\zeta^n}. \tag{1.15}$$

Thus, G_ζ^n denotes the inverse graph to the graph H_ζ^n. By definition:

The graph H_ζ^n and its inverse G_ζ^n are subsets of points in the real plane $\mathbb{R}^2$.

It is useful to illustrate the above definition of the inverse graph G_ζ^n before proceeding to the general case.