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A Categorical Approach to Imprimitivity Theorems for C^*-Dynamical Systems

Abstract

Imprimitivity theorems provide a fundamental tool for studying the representation theory and structure of crossed-product C^*-algebras. In this work, we show that the Imprimitivity Theorem for induced algebras, Green's Imprimitivity Theorem for actions of groups, and Mansfield's Imprimitivity Theorem for coactions of groups can all be viewed as natural equivalences between various crossed-product functors among certain equivariant categories.

The categories involved have C^*-algebras with actions or coactions (or both) of a fixed locally compact group G as their objects, and equivariant equivalence classes of right-Hilbert bimodules as their morphisms. Composition is given by the balanced tensor product of bimodules.

The functors involved arise from taking crossed products; restricting, inflating, and decomposing actions and coactions; inducing actions; and various combinations of these.

Several applications of this categorical approach are also presented, including some intriguing relationships between the Green and Mansfield bimodules, and between restriction and induction of representations.

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Contents

Introduction

Given a dynamical system (A, G, α) in which a locally compact group G acts by automorphisms of a C^*-algebra A, Mackey and Takesaki's *induction* process allows us to construct representations of (A, G, α) from representations of the system $(A, H, \alpha|_H)$ associated to any closed subgroup H of G. Much is known about induction: there are imprimitivity theorems which allow us to recognize induced representations, and the process is functorial with respect to intertwining operators.

In the modern framework of Rieffel, one introduces the crossed product $A \times_\alpha G$, which is a C^*-algebra encapsulating the representation theory of (A, G, α), and induces instead from $A \times_\alpha H$ to $A \times_\alpha G$; induction of representations from one C^*-algebra D to another C is achieved by tensoring the underlying Hilbert space with a Hilbert bimodule ${}_C X_D$, which has a D-valued inner product and in which the left action of C is by adjointable operators. An *imprimitivity theorem* tells us how to expand the left action of C to one of a larger algebra E in such a way that ${}_E X_D$ is an imprimitivity bimodule — that is, reversible. The theorem then says that a representation of C is equivalent to one induced from D if and only if there is a compatible representation of E.

Duality tells us how to recover a dynamical system (A, G, α) from its crossed product $A \times_\alpha G$. When G is abelian, the crossed product carries a canonical dual action $\widehat{\alpha}$ of the dual group $\widehat{G}$, and the Takesaki-Takai Duality Theorem says that the double dual system $\big((A \times_\alpha G) \times_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}}\big)$ is Morita equivalent to the original one. For nonabelian G, one has to use instead the dual coaction of G, and recover the system from the crossed product by this dual coaction. For duality to be a useful tool, one has to understand these coactions and their crossed products, and a good deal of progress has been made in the past 15 years. (An overview of this area has been provided in an Appendix; see also [**52**] for a recent survey.) Crucial for us is Mansfield's theory of induction for crossed products by coactions: he provides a Hilbert bimodule which allows us to induce representations from crossed products by quotient groups, and an imprimitivity theorem which characterizes these induced representations.

Induction and duality interact in deep and mysterious ways. One general principle appears to be that duality swaps induction of representations with restriction of representations. This is enormously appealing: restriction of representations (for example, passing from a representation U of G to the representation $U|_H$ of a subgroup H) is ostensibly a trivial process. Theorems making this induction-restriction duality precise have been proved, first for abelian groups in [**14**], and later for arbitrary groups in [**29, 18**]. We have gradually learned that it is best to prove such theorems by manipulating the Hilbert bimodules which implement the various induction and restriction processes; however, the bimodules involved are hard to work with — especially Mansfield's — and the results can safely be described as "technically challenging". To make things worse, applications frequently require that various isomorphisms and equivalences are equivariant, and one is continually having to construct compatible coactions on bimodules and check that they carry through complicated arguments. So it is definitely of interest to find a more systematic approach.

Our goal here is to provide such a systematic approach and to use it to complete our program of induction-restriction duality. We shall show that many of the key technical problems in this area amount to asking for functoriality of some construction or naturality of some equivalence between functors. Asking for equivalences to be equivariant amounts to asking for an equivalence in a different category, one which includes coactions or actions in its objects and morphisms. We have found that functoriality of the various crossed-product constructions encompasses many results of the kind "Morita equivalent systems have Morita equivalent crossed products", and naturality of the equivalences many results of the kind "induction is compatible with Morita equivalence".

To help see how our approach works, we consider one of our main theorems. It concerns the generalization of Green's Imprimitivity Theorem to crossed products of induced algebras, which is, loosely speaking, the analogue of the imprimitivity theorem for actions α of a subgroup H which do not extend to actions of G. The induced algebra $\mathrm{Ind}_H^G(A,\alpha)$ is a subalgebra of $C_b(G,A)$ which carries a left action τ of G by translation, and the generalization says that the crossed product $\mathrm{Ind}_H^G(A,\alpha)\times_\tau G$ is Morita equivalent to $A\times_\alpha H$. We shall prove that this equivalence is natural, and that it is equivariant for the dual coaction $\widehat{\tau}$ of G on $\mathrm{Ind}_H^G(A,\alpha)\times_\tau G$ and the inflation $\mathrm{Inf}\,\widehat{\alpha}$ to G of the dual coaction on $A\times_\alpha H$. To make this precise, we have to set up categories $\mathcal{C}$ of C^*-algebras, $\mathcal{A}(G)$ of dynamical systems (A,G,α), and $\mathcal{C}(G)$ of cosystems (A,G,δ) in which δ is a coaction of G on A. We then prove that $(A,G,\alpha)\mapsto(\mathrm{Ind}_H^G(A,\alpha)\times_\tau G,\widehat{\tau})$ and $(A,G,\alpha)\mapsto(A\times_\alpha H,\mathrm{Inf}\,\widehat{\alpha})$ are the object maps for functors from $\mathcal{A}(G)$ to $\mathcal{C}(G)$, so that it makes sense to say that they are naturally equivalent.

When we assert that, for example, $(A,G,\alpha)\mapsto(A\times_\alpha G,\widehat{\alpha})$ is a functor, we are completely ignoring the morphisms, and we cannot appreciate what naturality means until we deal with them too: a natural equivalence T between two functors $F,G\colon\mathcal{A}\to\mathcal{B}$ assigns to each object A of $\mathcal{A}$ an equivalence $T(A)\colon F(A)\to G(A)$ (that is, an invertible morphism $T(A)$ in the category $\mathcal{B}$) such that, for each morphism $\varphi\colon A\to B$ in $\mathcal{A}$, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{T(A)} & G(A) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle G(\varphi)} \\ F(B) & \xrightarrow[T(B)]{} & G(B) \end{array}$$

commutes in $\mathcal{B}$. In our categories, the morphisms will be based on Hilbert bimodules; in $\mathcal{A}(G)$, for example, a morphism from (A,G,α) to (B,G,β) will be given by a Hilbert bimodule ${}_AX_B$ with a compatible action γ of G. The composition of morphisms will be based on the balanced tensor product of bimodules, so that a diagram

$$\begin{array}{ccc} A & \xrightarrow{X} & B \\ {\scriptstyle Y}\downarrow & & \downarrow{\scriptstyle W} \\ C & \xrightarrow[Z]{} & D \end{array}$$

of Hilbert bimodules commutes if $Y \otimes_C Z \cong X \otimes_B W$ as Hilbert $A - D$ bimodules; in $\mathcal{A}(G)$ or $\mathcal{C}(G)$ this isomorphism has to be appropriately equivariant. The equivalences in these categories are the morphisms which are given by imprimitivity bimodules, so to prove that two of our functors F, G are naturally equivalent amounts to finding imprimitivity bimodules ${}_{F(A)}X(A)_{G(A)}$ such that

$$X(A) \otimes_{G(A)} G(Y) \cong F(Y) \otimes_{F(B)} X(B)$$

as Hilbert $F(A) - G(B)$ bimodules for each Hilbert bimodule ${}_AY_B$. The modules are the usual ones, but many of the details needed to establish these isomorphisms and their properties are new.

This paper, like any other in which coactions appear, involves some gritty technical arguments. We will therefore begin by outlining the main new issues which we face in this program, and how we have dealt with them. Those who are interested in seeing how the categorical ideas impact when there are no coactions around are encouraged to read our previous paper [**17**] first. Indeed, this might help even those who are already coaction-compliant!

Outline

We begin in Chapter 1 with a detailed discussion of the Hilbert bimodules on which our morphisms are based. The axioms are intrinsically asymmetric; to see why, note that a homomorphism $\varphi\colon A \to B$ gives B the structure of an A-module, but not the other way round. Our modules ${}_AX_B$ will be right Hilbert B-modules with a left action of A given by a nondegenerate homomorphism κ of A into the C^*-algebra $\mathcal{L}(X_B)$ of adjointable operators on X. As in [**29**], we shall call these *right-Hilbert bimodules* to emphasize that the Hilbert-module structure is on the right; we have stuck with this name because the alternatives (C^*-correspondences or Hilbert bimodules) do not carry the same sense of direction. The theory of right-Hilbert bimodules is similar to that of imprimitivity bimodules, but there seem to be enough subtle differences to warrant a detailed discussion.

The first section contains the basic facts about multiplier bimodules and homomorphisms between bimodules. These are used repeatedly: a coaction on a bimodule X, for example, is by definition a homomorphism of X into the multiplier bimodule $M(X \otimes C^*(G))$. Our treatment is similar to that of imprimitivity bimodules in [**20**]. Section 1.1.3 is about the balanced tensor products which are used to define the composition of morphisms; we need to know in particular how this process extends to multipliers. We also discuss external tensor products, which are crucial for the definition of coactions on Hilbert bimodules. The last section of Chapter 1 is about linking algebras. These are used primarily as a technical tool in the proofs of naturality (an idea lifted from [**21**], and expounded in an easier setting in [**17**]).

In Chapter 2 we describe the categories in which we work. The basic category $\mathcal{C}$ of C^*-algebras appears in [**17**]; we review the main facts in Section 2.1. The objects are C^*-algebras and the morphisms from A to B are the isomorphism classes of right-Hilbert $A - B$ bimodules: we have to pass to isomorphism classes to ensure that the composition law $[_CY_B] \circ [_AX_B] = [_A(X \otimes_C Y)_B]$ has the required properties. The other categories $\mathcal{A}(G)$, $\mathcal{C}(G)$ and $\mathcal{AC}(G)$ are associated to a fixed locally compact group G, and are obtained by adding, respectively, actions of G, coactions of G, and both actions and coactions to the objects and morphisms of $\mathcal{C}$. Adding actions is relatively routine, but (as will be no surprise to those familiar with them) adding coactions is a little harder. (Coactions on Hilbert bimodules first appeared in [**2**].) We show that in each of these categories, the equivalences (that is, the invertible morphisms) are the morphisms in which the underlying bimodules are imprimitivity bimodules, and then that every morphism is a composition of a morphism coming from a nondegenerate homomorphism $\varphi\colon A \to M(C)$ and a morphism based on an imprimitivity bimodule $_CX_B$.

In Chapter 3 we show that the various crossed products appearing in our theorems define functors between appropriate categories. There are two main problems. The first is to define suitable crossed products. We are interested here in coactions and nonabelian duality, which is basically a theory about reduced crossed products, so we have decided to give in gracefully and use reduced crossed products throughout. (This is definitely a choice: we have already proved the naturality of Green's Imprimitivity Theorem for full crossed products in [**17**], and providing we were willing to omit all statements about the coactions, we could presumably do the same here.) But because the objects in our categories are C^*-algebras rather than isomorphism classes of C^*-algebras, it is important that we don't just *choose* a regular representation willy-nilly. So we shall discuss a specific realization of the reduced product. The second main problem is to define crossed products of the Hilbert bimodules which define the morphisms. We do this differently for actions and coactions; for actions we make heavy use of the convenience of C_c-functions, and for coactions we realize the crossed product inside a certain multiplier bimodule. For imprimitivity bimodules, it is handy to recognize that if $L(X)$ is the linking algebra of X, then the bimodule crossed product $X \times G$ embeds as the top right corner of $L(X) \times G$, and we have the important relation $L(X) \times G = L(X \times G)$ almost by definition. We should mention that defining these crossed products and establishing their properties has been done before; see [**2**], [**7**], [**6**], [**20**], and [**30**].

We gather all the necessary functors in Chapter 3; even though some are easy, it is convenient to deal with them all at once. The key difficulty is the same in each case: it is not obvious that crossed products preserve composition. This amounts to proving things like

$$(X \otimes_B Y) \times G \cong (X \times G) \otimes_{B \times G} (Y \times G),$$

and again our techniques are different for actions and coactions.

Our main theorems are in Chapter 4. We have already discussed the first, which is about crossed products of induced algebras, and which we prove in Section 4.1. The proofs of this and our other main theorems follow the same general pattern. We factor each morphism $_AX_B$ as a composition of a nondegenerate homomorphism

$\varphi\colon A \to M(C)$ and an imprimitivity bimodule ${}_C Y_B$. To prove that

$$\begin{array}{ccc} F(A) & \xrightarrow{T(A)} & G(A) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle G(\varphi)} \\ F(C) & \xrightarrow[T(C)]{} & G(C) \end{array}$$

commutes, we extend the homomorphisms $(F(\varphi), G(\varphi))$ to a homomorphism of imprimitivity bimodules $T(A) \to M(T(C))$, and use a general lemma which says this suffices. To prove that

$$\begin{array}{ccc} F(C) & \xrightarrow{T(C)} & G(C) \\ {\scriptstyle F(Y)}\downarrow & & \downarrow{\scriptstyle G(Y)} \\ F(B) & \xrightarrow[T(B)]{} & G(B) \end{array}$$

commutes, we realize $T(C)$ and $T(D)$ as the diagonal corners in an imprimitivity bimodule Z over the linking algebras $L(F(Y))$ and $L(G(Y))$, and use a general lemma from [**21**] which identifies both $F(Y) \otimes_{F(B)} T(B)$ and $T(C) \otimes_{G(C)} G(Y)$ with the top off-diagonal corner in Z. The hard part in both halves is to build the compatible coaction.

It may be known that this theorem about crossed products of induced algebras is a generalization of Green's Imprimitivity Theorem, but it does not appear to be well-documented. We therefore give a careful derivation, which could be of some independent interest (see the discussion preceding Theorem B.3 in Appendix B). We then use this to deduce our second main theorem, which is a natural and equivariant version of the Imprimitivity Theorem itself. There are many possible variations on this theme, depending on choices of full and reduced crossed products and on whether or not the subgroup is normal. Here we have already decided to use reduced crossed products, and we have further chosen to discuss what happens for normal subgroups. We have made this choice because in this case there are several more actions and coactions in play, and the theorem has something to say about all of them. To see what is happening here, recall that if N is normal, we can view the imprimitivity algebra $(A \otimes C_0(G/N)) \times_{\alpha\otimes\tau} G$ in Green's theorem as the crossed product $(A \times_\alpha G) \times_{\widehat{\alpha}|} G/N$ by the restriction of the dual coaction; thus this imprimitivity algebra carries a dual action $(\widehat{\alpha}|)\widehat{}$ of G/N as well as a dual coaction $(\alpha \otimes \tau)\widehat{}$ of G. Our theorem says that Green's imprimitivity bimodule matches $(\widehat{\alpha}|)\widehat{}$ with the so-called decomposition action of G on $A \times_\alpha N$ and $(\alpha \otimes \tau)\widehat{}$ with the inflation to G of the dual coaction of N on $A \times_\alpha N$. This observation seems to be new. Indeed, we believe that the equivariance and the naturality are both potentially important new pieces of information about Green's theorem.

Our third main theorem is a version of Mansfield's Imprimitivity Theorem. This has all the same features as the version of Green's theorem which we have just discussed: Mansfield's Morita equivalence of $(A \times_\delta G) \times_{\widehat{\delta}} N$ with $A \times_{\delta|} G/N$ is natural and equivariant for canonical actions and coactions on the crossed products. For this theorem, the difficult part of the proof is establishing the naturality with respect to ordinary homomorphisms $\varphi\colon A \to M(C)$; we have to work hard to build compatible homomorphisms on Mansfield's bimodule.

In Chapter 5 we give some applications to our motivating problem of understanding the relationships between induction and duality. In Section 5.1, we uncover some new and very intriguing relationships between Green and Mansfield induction. Important special cases of these results say that the Green bimodules $X^G_{\{e\}}(A)$ and Mansfield bimodules $Y^G_{G/G}(A)$ are in duality:

$$X^G_{\{e\}}(A) \times G \cong Y^G_{G/G}(A \times G)$$

and

$$X^G_{\{e\}}(A \times G) \cong Y^G_{G/G}(A) \times G.$$

Results of this type require several applications of our main theorems, and it is vital that we know everything is appropriately equivariant. Our main new application to induction-restriction duality is Theorem 5.16, which completes the program of [**14, 29, 18**] by handling the restriction of representations from $A \times_\alpha G$ to $A \times_{\alpha|} N$. We close with a new application of linking-algebra techniques to the Symmetric Imprimitivity Theorem of [**51**].

Since this project is intrinsically involved with nonabelian duality, we have necessarily made heavy use of coactions and their crossed products. There are several different sets of definitions available: the subject is stabilizing, but some key questions of a fundamental nature remain unresolved, and hence this is taking longer than one might have wished. So we have included as an appendix a survey of the area, which outlines what we believe to be the most satisfactory approach and describes how this approach relates to the others in the literature.

A second appendix collects the precise versions of the imprimitivity theorems we need; various formulations appear in the literature, so we felt it would be handy to record exactly what we want.

Finally, the third appendix contains some technical results on function spaces with values in locally convex spaces which are used throughout the text to construct multipliers of bimodules. In applications, the locally convex spaces will be multiplier algebras or bimodules with the strict topology: we need to know, for example, that strictly continuous functions of compact support from G to $M(X)$ define multipliers of $X \times G$, and that they do so in an orderly fashion.

Epilogue

Although this paper has turned out much longer than we intended, we have made all sorts of simplifying assumptions to keep the length down, and these are probably logically unnecessary. First of all, we have deliberately excised twisted crossed products, though some residual traces remain in the presence of the decomposition actions and coactions. Any serious application of these ideas to the Mackey machine — which was, after all, our original motivation [**14**] — will require that we can handle twisted crossed products. Second, we have used reduced crossed products throughout. For our present applications involving nonabelian duality and crossed products by coactions, this makes sense: the current duality theorems all factor through the reduced crossed product. But for applications to ordinary crossed products this is not necessarily desirable, and there are surely versions of Theorem 4.1 and Theorem 4.2 for full crossed products. We have already described a version of Green's theorem in [**17**], but we neglected questions of equivariance there. Third, we have considered only some of the important Morita equivalences. The others, such as the Symmetric Imprimitivity Theorem and the Stabilization

Trick for twisted crossed products, should be natural too. (Working in the context of general locally compact quantum groups was not even an issue, since there are currently no imprimitivity theorems available in that generality!)

On the other hand, we have taken the liberty of treating actions separately from coactions — rather than viewing actions of G as coactions of $C_0(G)$ — although this would have led to a much shorter exposition. Our main reasons for this are that we think that actions are much easier to understand than coactions, and that we feel there may be more general interest in the action case than in the coaction case.

We hope that we have given convincing evidence that issues involving functoriality, naturality and equivariance are likely to occur frequently in our subject, and that it will pay for us get in the habit of dealing with them as we go. We also hope that we have made a few other points along the way: our view of induced C^*-algebras as an obstruction to imprimitivity, our heavy use of linking-algebra techniques to identify imprimitivity bimodules, and the seemingly deep and strange relations between induction and duality, should all have applications elsewhere. For instance, the strong connection between these ideas and equivariant KK-theory is well-documented in [**2**] and [**30**]. Several of the ideas are also present in the approach of [**9**] and [**8**] towards a Mackey machine for the Baum-Connes conjecture, and are applied to the Connes-Kasparov conjecture in [**10**].

CHAPTER 1

Right-Hilbert Bimodules

In this chapter we gather together the basic theory of right-Hilbert bimodules. We start with the basic definitions and some important notation which shall be used throughout this work.

1.1. Right-Hilbert bimodules and partial imprimitivity bimodules

Let B be a C^*-algebra. Recall that a *Hilbert B-module* is a vector space X which is a right B-module equipped with a positive definite B-valued sesquilinear form $\langle\cdot,\cdot\rangle_B$ satisfying

(1.1) $\langle x, y\cdot b\rangle_B = \langle x,y\rangle_B b$ and $\langle x,y\rangle_B^* = \langle y,x\rangle_B$ for all $x,y\in X, b\in B$,

and which is complete in the norm $\|x\| = \|\langle x,x\rangle_B\|^{1/2}$. Our primary reference for Hilbert modules is [**54**], and a secondary reference is [**33**]. Some notational conventions: we often omit the dot $(\cdot)$ when writing module actions; and in general, if $(u,v)\mapsto uv\colon U\times V\to W$ is a pairing among vector spaces, then for $P\subseteq U$ and $Q\subseteq V$ we write PQ to mean the *linear span* of the set $\{uv \mid u\in P, v\in Q\}$.

DEFINITION 1.1. Let A and B be C^*-algebras. A *right-Hilbert A–B bimodule* is a Hilbert B-module X which is also a nondegenerate left A-module (*i.e.*, $AX = X$) satisfying

(1.2a) $$a\cdot(x\cdot b) = (a\cdot x)\cdot b \quad \text{and}$$

(1.2b) $$\langle a\cdot x, y\rangle_B = \langle x, a^*\cdot y\rangle_B$$

for all $a\in A$, $x,y\in X$, and $b\in B$. We write ${}_AX_B$ to indicate all the data, and we call X *full* if it is full as a Hilbert B-module, *i.e.*, $\overline{\langle X,X\rangle}_B = B$. In general, if X is not full, we shall write B_X for the closed ideal $\overline{\langle X,X\rangle}_B\subseteq B$, and we call B_X the *range* of the inner product on X.

REMARK 1.2. (1) In recent years, objects very similar to right-Hilbert bimodules have been introduced into the literature: for example, the $A-B$ correspondences of [**39**]. In many cases (as in [**39**]), the left module action is permitted to be degenerate; we require it to be nondegenerate so that we can extend it to the multiplier algebra $M(A)$ (see below).

(2) Note that if X is an $A-B$ correspondence, then $\overline{AX}$ is a closed $A-B$ sub-bimodule of X, and therefore becomes a right-Hilbert $A-B$ bimodule. In fact we have $\overline{AX} = AX = \{ax \mid a\in A, x\in X\}$, since it follows from Cohen's factorization theorem that $\overline{AX} = A\overline{AX}\subseteq AX$ (we refer to [**54**, Proposition 2.33] for a statement and an easy proof of Cohen's factorization theorem in the case where A is a C^*-algebra). More generally, a similar application of Cohen's theorem implies that for any C^*-subalgebras C and D of A and B, respectively, we have $\overline{CX} = \{cx \mid c\in C, x\in X\}$ and $\overline{XD} = \{xd \mid x\in X, d\in D\}$.

EXAMPLE 1.3. If B is a C^*-algebra, then B becomes a full right-Hilbert $B-B$ bimodule in a natural way by putting

$$a \cdot b \cdot c = abc \quad \text{and} \quad \langle a, b\rangle_B = a^*b \qquad \text{for } a, b, c \in B.$$

If $\varphi\colon A \to M(B)$ is a nondegenerate C^*-algebra homomorphism, then B becomes a full right-Hilbert $A-B$ bimodule with left action given by

$$a \cdot b = \varphi(a)b.$$

More generally, if $\varphi\colon A \to M(B)$ is an arbitrary (possibly degenerate) $*$-homomorphism, then B becomes an $A-B$ correspondence and, therefore, $X = \varphi(A)B$ is a right-Hilbert $A-B$ bimodule. We call a right-Hilbert bimodule ${}_AX_B$ arising in this way *standard.* If $\varphi\colon A \to M(B)$ is nondegenerate, *i.e.*, if $X = \varphi(A)B = B$, then we say that ${}_AB_B$ is a *nondegenerate* standard right-Hilbert bimodule.

REMARK 1.4. It is clear that a nondegenerate standard right-Hilbert bimodule ${}_AB_B$ is full. The converse is not true in general. To see an example let $B = M_2(\mathbb{C})$ and let $\varphi\colon \mathbb{C} \to M_2(\mathbb{C}); \varphi(\lambda) = \left(\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right)$. Then $M_2(\mathbb{C})\varphi(\mathbb{C})M_2(\mathbb{C}) = M_2(\mathbb{C})$ and $X = \varphi(\mathbb{C})M_2(\mathbb{C}) \cong \mathbb{C}^2$ is a full right-Hilbert $\mathbb{C} - M_2(\mathbb{C})$ bimodule, but φ is degenerate.

If X and Y are Hilbert B-modules, $\mathcal{L}_B(X, Y)$ denotes the set of maps $T\colon X \to Y$ which are adjointable in the sense that there exists $T^*\colon Y \to X$ such that

$$\langle Tx, y\rangle_B = \langle x, T^*y\rangle_B \qquad \text{for all } x \in X, y \in Y.$$

Such T are automatically bounded and B-linear [**54**, Lemma 2.18]. The notation is shortened to $\mathcal{L}(X, Y)$ if B is understood, and $\mathcal{L}_B(X)$ (or just $\mathcal{L}(X)$) if $X = Y$. In the latter case $\mathcal{L}(X)$ is a C^*-algebra with the operator norm $\|T\| = \sup\{\|Tx\| \mid \|x\| \leq 1\}$ [**54**, Proposition 2.1].

Now, if ${}_AX_B$ is a right-Hilbert $A-B$ bimodule then for each $a \in A$ the map $x \mapsto a \cdot x$ is adjointable (consequently the associativity condition (1.2a) is redundant), so we get a homomorphism $\kappa\colon A \to \mathcal{L}_B(X)$ such that

$$\kappa(a)x = a \cdot x,$$

and which is nondegenerate in the sense that $\kappa(A)X = X$. Conversely, every right-Hilbert bimodule arises in this way: If X is a Hilbert B-module and $\kappa\colon A \to \mathcal{L}(X)$ is a nondegenerate homomorphism, then X becomes a right-Hilbert $A-B$ bimodule via

$$a \cdot x = \kappa(a)x.$$

Thus a right-Hilbert $A-B$ bimodule is nothing more nor less than a Hilbert B-module X together with a nondegenerate homomorphism $A \to \mathcal{L}(X)$.

If X and Y are Hilbert B-modules, $\mathcal{K}(X, Y)$ denotes the *compact operators* from X to Y: by definition, it is the closed span in $\mathcal{L}(X, Y)$ of the maps $z \mapsto y\langle x, z\rangle_B$ for $x \in X$ and $y \in Y$. $\mathcal{K}(X) = \mathcal{K}(X, X)$ is a closed ideal in $\mathcal{L}(X)$, and in fact $\mathcal{L}(X) \cong M(\mathcal{K}(X))$ [**54**, Corollary 2.54]. In particular, if X is a Hilbert B-module, then the formula

$$(1.3) \qquad {}_{\mathcal{K}(X)}\langle x, y\rangle\, z = x\,\langle y, z\rangle_B$$

defines a full $\mathcal{K}(X)$-valued inner product on X, which gives X the structure of a *left* Hilbert $\mathcal{K}(X)$-module. Then B acts via adjointable operators on the right of ${}_{\mathcal{K}(X)}X$, and $B_X = \overline{\langle X, X\rangle}_B$ identifies with the compact operators of the left Hilbert $\mathcal{K}(X)$-module ${}_{\mathcal{K}(X)}X$.