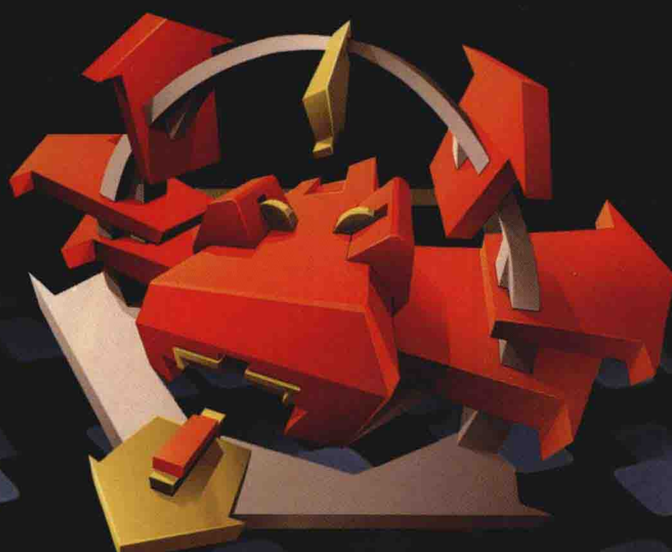
Differential Equations and Applications

Recent Advances

Editor

P. Prakash



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Preface

Theory of nonlinear differential equations play an important role in science, engineering and social sciences since, many problems of these branches have been solved by the use of its methods and properties. Also, various powerful methods have been used to explore different possible solutions of the problem considered. Recently, nonlinear differential equations of fractional order have been proved to be a valuable tool for the modeling of many areas and the use of fractional order derivatives and integrals gives better approximation than that of integer-order. Moreover, several numerical methods were employed for the numerical treatment of the nonlinear problems.

In recognition of Srinivasa Ramnaujan's contribution to Mathematics, the Government of India declared that Ramanujan's birthday to be celebrated as 'National Mathematics Day' on 22nd December of every year. Also, Government of India declared the year 2012 as the 'National Mathematical Year' to commemorate his 125th Birth Anniversary.

In view of the above, Department of Mathematics, Periyar University, Salem organized this conference to exchange recent developments, discuss issues of common concern, establish contacts, and gather information that would be of use to those in the Mathematics community. The conference aims to present a broad and interdisciplinary overview of the current, state-of-the-art methods and techniques for characterizing partial differential equations.

There were 10 expository lectures by eminent Mathematicians in the field which mainly focused on Partial Differential Equations with emphasis on parabolic and hyperbolic problems one of the thrust areas of current research in the world. Out of 41 papers presented the referees have recommended only 25 papers for publication. The Department of Mathematics expresses its sincere thanks and gratitude to all the referees and also to the various organizations for their support in organizing this conference.

On behalf of the organizing committee the convener acknowledge, with gratitude, the generous financial support provided by the National Board for Higher Mathematics (Department of Atomic Energy), Mumbai, Department of Science and Technology and Council for Scientific and Industrial Research, New Delhi.

P. Prakash

Contents

Differential Equations and Applications: Recent Advances
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Analytical and numerical solutions of an one dimensional fractional sub-diffusion equation with Neumann boundary conditions

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Abstract: In this paper, one-dimensional fractional sub-diffusion equation (FSDE) with Neumann boundary conditions is studied both analytically and numerically. An implicit difference approximation scheme (IDAS) is developed. We analyze the local truncation error and discuss the stability using the energy method. Then we prove the convergence. Numerical results are provided to verify the accuracy and efficiency of the proposed algorithm.

1 Introduction

Fractional differential equations have been the focus of many studies due to their frequent occurrence in various fields such as physics, chemistry, viscoelasticity, fluid mechanics, biology, acoustics, control theory and psychology etc. A realistic model of a physical phenomenon having dependence not only on the instant time, but also on the previous time history can be successfully achieved by using fractional calculus. Such a calculus can be named as non-integer order calculus and the subject can be traced back to the genesis of integer order differential calculus itself. Though Leibniz made some remarks on the meaning and possibility of fractional derivatives of order 1/2 in the late seventeenth century in a rigorous series of papers from 1832 to 1837, where he defined for the first time an operator of fractional integration. Today fractional calculus extends the derivative and anti-derivative operations of differential and integral calculus from non-integer orders to the entire complex plane. There are several approaches to the generalization of the notion of differentiation to fractional orders, for example, Riemann-Liouville, Grunwald-Letnikov, Caputo and generalized functions approach [8–11]. Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions which have not been given physically meaningful explanation. Caputo introduced an alternative definition which has the advantage of defining integer order initial conditions for fractional order differential equations. Unlike the Riemann-Liouville approach which derives its definition from repeated integration, the Grunwald-Letnikov formulation approaches the problem from the derivative side. This is mostly used in numerical algorithms.

Nowadays fractional diffusion equations play an important role in modeling anomalous diffusion and sub-diffusion systems, in the description of fractional random walk, unification of diffusion and wave propagation phenomena see for example the reviews in [1–3, 5–7, 14, 15] and the references there-in. Liu has carried out so many works on the finite difference method of fractional diffusion equations [2, 3, 15]. Langlands and Henry [5] investigated the accuracy and stability of an implicit numerical scheme for solving a fractional diffusion equation with zero flux boundary condition.

The purpose of this paper is to solve the fractional sub-diffusion equation with Neumann boundary conditions using the Implicit Difference Approximation Scheme and give the stability and convergence analysis. The model problem considered here is

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left[K_\gamma \frac{\partial^2 u(x,t)}{\partial x^2}\right] + f(x,t), \quad 0 \le x \le L, \quad 0 < t \le T. \tag{1}$$

The initial condition for (1) is
$$u(x,0) = w(x), \tag{2}$$

and the Neumann boundary condition for (1) is

$$u_x(0,t) = \phi(t), \quad u_x(L,t) = \xi(t), \tag{3}$$

where K_γ is the generalized diffusion constant, $0 < \gamma < 1$, ${}_0D_t^{1-\gamma}u(x,t)$ denotes Riemann-Liouville fractional derivative of order $1-\gamma$ of the function $u(x,t)$ defined by [8–10], that is,

$${}_0D_t^{1-\gamma}u(x,t) = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{1-\gamma}}d\tau, \tag{4}$$

and $f(x,t), w(x), \phi(t), \xi(t)$ are known smooth functions. We assume that the equations (1)-(3) have a unique solution $u(x,t) \in C_{x,t}^{2,1}([0,L]\times[0,T])$. Then (1) can be put in the equivalent form [3]

$${}_0D_t^{\gamma}[u(x,t) - u(x,0)] = K_\gamma \frac{\partial^2 u(x,t)}{\partial x^2} + g(x,t), \tag{5}$$

where $g(x,t) = {}_0D_t^{\gamma-1}f(x,t)$.

The remainder of the paper is organized as follows: in section 2, the analytical solution of the fractional sub-diffusion equation (FSDE) is given. In section 3, we present an implicit difference approximation scheme (IDAS). We approximate the first and second-order space derivatives by the central difference, then use the Grunwald-Letnikov discretization for the approximation of the time fractional derivative. In section 4, the matrix form of the difference scheme is given and the solvability for the linear system of equations is discussed. In section 5, we give the local truncation error, investigate the stability by energy method; we prove that the scheme is unconditionally stable for all γ in the range $0 < \gamma < 1$ and derive the global accuracy and prove the convergence of the scheme. Finally some numerical results are provided in section 6 which are in agreement with our theoretical analysis. The paper ends with a brief conclusion section. In this paper, we use the "empty sum" convention $\sum_{l=k}^{n} u^l = 0$ for $n < k$.

2 Analytical solution of TFDE

Using the relationship between Caputo fractional derivative and Riemann-Liouville fractional derivative, (5) can be rewritten as

$${}_0^CD_t^{\gamma}u(x,t) = K_\gamma \frac{\partial^2 u(x,t)}{\partial x^2} + g(x,t),$$

where ${}_0^CD_t^{\gamma}u(x,t)$ denotes the caputo fractional derivative of order γ of the function $u(x,t)$ defined by [10]

$${}_0^CD_t^{1-\gamma}u(x,t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\partial u(x,\tau)}{\partial t}\frac{d\tau}{(t-\tau)^{\gamma}},$$

Applying finite cosine transform [4] with respect to the spatial variable x, we have

$$ {}_0^C D_t^\gamma [U(n,t)] - \frac{t^{-\gamma}}{\Gamma[1-\gamma]} U(n,0) = -K_\gamma (na)^2 U(n,t) + H(t) + G(n,t), $$

where $a = \frac{\pi}{L}$, $H(t) = \frac{-2K_\gamma}{L}[(-1)^{n+1}u_x(L,t) + u_x(0,t)]$ and $U(n,t)$, $U(n,0)$, $G(n,t)$ are finite cosine transform of $u(x,t)$, $u(x,0)$, and $g(x,t)$ respectively. Applying Laplace transform with respect to the time variable t, we have

$$ \bar{U}(n,s) = \frac{s^{\gamma-1}}{s^\gamma + K_\gamma (na)^2} U(n,0) + \frac{\bar{H}(s) + \bar{G}(n,s)}{s^\gamma + K_\gamma (na)^2}. $$

Applying inverse Laplace transform, we have

$$ U(n,t) = E_{\gamma,1}[-K_\gamma (na)^2 t^\gamma] U(n,0) + (H(t) + G(n,t)) * t^{\gamma-1} E_{\gamma,\gamma}[-K_\gamma (na)^2 t^\gamma], \tag{6} $$

where $E_{\alpha,\beta}$ is the Mittag Leffler function defined by $E_{\alpha,\beta} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)}$ and $f(t) * g(t)$ is the convolution of these functions and is defined by $f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau$. Applying inverse finite cosine transform, we have

$$ u(x,t) = \frac{U(0,t)}{2} + \sum_{n=1}^{\infty} U(n,t) \cos(nax), $$

where $U(n,t)$ is given in (6).

3 The implicit difference approximation scheme

3.1 Partition and the solution vector

We introduce a uniform grid of points (x_j, t_k), with $x_j = jh$, $j = 0,1,2,...m$, and $t_k = k\tau$, $k = 0,1,2,...n$, where m and n are positive integers, $h = L/m$ is the mesh-width in x and $\tau = T/n$, the time step. Let $\Omega_h = \{x_j | 0 \le j \le m\}$ and $\Omega_\tau = \{t_k | 0 \le k \le n\}$. The exact solution u of FSDE at the point (x_j, t_k) is denoted by u_j^k and the corresponding solution vector is denoted by $u^k = u(t_k) = (u_0^k, u_1^k, ..., u_m^k)^T$. The exact solution of an approximating difference scheme at the same point will be denoted by U_j^k and the corresponding solution vector is denoted by $U^k = U(t_k) = (U_0^k, U_1^k, ..., U_m^k)^T$.

3.2 Derivation of numerical scheme

Using the relationship between the Grunwald - Letnikov formula and the Riemann - Liouville fractional derivatives [9, 10], we can approximate the fractional derivative by

$$ {}_0D_t^\gamma [u_j^k - u_j^0] = \tau^{-\gamma} \sum_{i=0}^{k} \lambda_i [u_j^{k-i} - u_j^0] + o(\tau), \tag{7} $$

where

$$ \lambda_i = (-1)^i \begin{pmatrix} \gamma \\ i \end{pmatrix}, \quad i = 0,1,...,k. \tag{8} $$

We use the first and second order central difference schemes for the first and second order spatial derivatives respectively [12]:

$$ \frac{\partial u_j^k}{\partial x} = \frac{u_{j+1}^k - u_{j-1}^k}{2h} + o(h^2), \tag{9} $$

$$ \frac{\partial^2 u_j^k}{\partial x^2} = \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2} + o(h^2). \tag{10} $$

3.3 An implicit difference approximation scheme (IDAS)

The initial boundary value problem (2), (3) and (5) can be approximated by the following implicit difference approximation scheme. For $1 \le k \le n$, we have

$$\begin{aligned}
\tau^{-\gamma}\left[U_j^k + \sum_{i=1}^{k-1}\lambda_i U_j^{k-i} - \sum_{i=0}^{k-1}\lambda_i U_j^0\right] &= \frac{K_\gamma}{h^2}[U_{j+1}^k - 2U_j^k + U_{j-1}^k] + g_j^k, \qquad (11)\\
&\qquad 0 \le j \le m,\\
\frac{U_1^k - U_{-1}^k}{2h} &= \phi(t_k), \qquad (12)\\
\frac{U_{m+1}^k - U_{m-1}^k}{2h} &= \xi(t_k), \qquad (13)\\
U_j^0 &= w(x_j), \quad 0 \le j \le m,
\end{aligned}$$

where $g_j^k = g(x_j, t_k)$. Substitute (12) and (13) in (11) when j=0 and j=m respectively, to have

$$\begin{aligned}
\tau^{-\gamma}\left[U_0^k + \sum_{i=1}^{k-1}\lambda_i U_0^{k-i} - \sum_{i=0}^{k-1}\lambda_i U_0^0\right] &= \frac{2K_\gamma}{h^2}[U_1^k - U_0^k] - \frac{2K_\gamma}{h}\phi(t_k) + g_0^k,\\
\tau^{-\gamma}\left[U_j^k + \sum_{i=1}^{k-1}\lambda_i U_j^{k-i} - \sum_{i=0}^{k-1}\lambda_i U_j^0\right] &= \frac{K_\gamma}{h^2}[U_{j+1}^k - 2U_j^k + U_{j-1}^k] + g_j^k,\\
&\qquad 1 \le j \le m-1,\\
\tau^{-\gamma}\left[U_m^k + \sum_{i=1}^{k-1}\lambda_i U_m^{k-i} - \sum_{i=0}^{k-1}\lambda_i U_m^0\right] &= \frac{2K_\gamma}{h^2}[U_{m-1}^k - U_m^k] + \frac{2K_\gamma}{h}\xi(t_k) + g_m^k,\\
U_j^0 &= w(x_j), \quad 0 \le j \le m. \qquad (14)
\end{aligned}$$

Introduce the scaling parameter $\mu = K_\gamma \tau^\gamma / h^2$ and after rearranging the terms, we have

$$\begin{aligned}
(1+2\mu)U_0^k &= -\sum_{i=1}^{k-1}\lambda_i U_0^{k-i} + \sum_{i=0}^{k-1}\lambda_i U_0^0 + 2\mu U_1^k - 2h\mu\phi(t_k) + \tau^\gamma g_0^k,\\
(1+2\mu)U_j^k &= -\sum_{i=1}^{k-1}\lambda_i U_j^{k-i} + \sum_{i=0}^{k-1}\lambda_i U_j^0 + \mu[U_{j+1}^k + U_{j-1}^k] + \tau^\gamma g_j^k,\\
&\qquad 1 \le j \le m-1,\\
(1+2\mu)U_m^k &= -\sum_{i=1}^{k-1}\lambda_i U_m^{k-i} + \sum_{i=0}^{k-1}\lambda_i U_m^0 + 2\mu U_{m-1}^k + 2h\mu\xi(t_k) + \tau^\gamma g_m^k,\\
U_j^0 &= w(x_j), \quad 0 \le j \le m, \qquad (15)
\end{aligned}$$

or

$$\begin{aligned}
(1+2\mu)U_0^k - 2\mu U_1^k &= -\sum_{i=1}^{k-1}\lambda_i U_0^{k-i} + \sum_{i=0}^{k-1}\lambda_i U_0^0 - 2h\mu\phi(t_k) + \tau^\gamma g_0^k,\\
-\mu U_{j-1}^k + (1+2\mu)U_j^k - \mu U_{j+1}^k &= -\sum_{i=1}^{k-1}\lambda_i U_j^{k-i} + \sum_{i=0}^{k-1}\lambda_i U_j^0 + \tau^\gamma g_j^k,\\
&\qquad 1 \le j \le m-1,
\end{aligned}$$

$$
\begin{aligned}
-2\mu U_{m-1}^k + (1+2\mu)U_m^k &= -\sum_{i=1}^{k-1}\lambda_i U_m^{k-i} + \sum_{i=0}^{k-1}\lambda_i U_m^0 + 2h\mu\xi(t_k) + \tau^\gamma g_m^k, \\
U_j^0 &= w(x_j), \quad 0 \le j \le m. \qquad (16)
\end{aligned}
$$

4 Matrix form of the IDAS

We give the matrix form of the IDAS (15) or (16) by

$$
AU^k = -\sum_{i=1}^{k-1}\lambda_i U^{k-i} + \sum_{i=0}^{k-1}\lambda_i U^0 + G^k, \quad 1 \le k \le n,
$$

where

$$
A = \begin{pmatrix}
1+2\mu & -2\mu & & & \\
-\mu & 1+2\mu & -\mu & & \\
 & \ddots & \ddots & \ddots & \\
 & & -\mu & 1+2\mu & -\mu \\
 & & & -2\mu & 1+2\mu
\end{pmatrix}_{(m+1)\times(m+1)},
$$

$$
G^k = \begin{pmatrix}
-2h\mu\phi(t_k) + \tau^\gamma g_0^k \\
\tau^\gamma g_1^k \\
\vdots \\
\tau^\gamma g_{m-1}^k \\
2h\mu\xi(t_k) + \tau^\gamma g_m^k
\end{pmatrix}_{(m+1)\times 1}, \quad 1 \le k \le n.
$$

For the solvability of the scheme, we have

Theorem 4.1. *The difference equations (15) or (16) has a unique solution.*

Proof. Because, for any $\mu = K_\gamma \frac{\tau^\gamma}{h^2} > 0$, the coefficient matrix A for the difference equation is strictly diagonally dominant. Consequently the matrix A is non singular, thus is invertible. Hence completes the proof of the theorem. □

5 Theoretical analysis of the IDAS

5.1 The local truncation error

The local truncation error of IDAS (15) or (16) is

$$
\begin{aligned}
R_j^k &= \tau^{-\gamma}\left[u_j^k + \sum_{i=1}^{k-1}\lambda_i u_j^{k-i} - \sum_{i=0}^{k-1}\lambda_i u_j^0\right] - \frac{K_\gamma}{h^2}[u_{j+1}^k - 2u_j^k + u_{j-1}^k] - g_j^k \\
&= \tau^{-\gamma}\left[u_j^k + \sum_{i=1}^{k-1}\lambda_i u_j^{k-i} - \sum_{i=0}^{k-1}\lambda_i u_j^0\right] - {}_0D_t^\gamma[u_j^k - u_0^k] \\
&\quad + K_\gamma\left[\frac{\partial^2 u_j^k}{\partial x^2} - \frac{1}{h^2}(u_{j+1}^k - 2u_j^k + u_{j-1}^k)\right] \\
&= o(\tau + h^2), \quad 1 \le j \le m-1, \quad 1 \le k \le n. \qquad (17)
\end{aligned}
$$

Using the Taylor series expansion of u_1^k about the point (x_0, t_k), we get

$$\begin{aligned} R_0^k &= \tau^{-\gamma}\left[u_0^k + \sum_{i=1}^{k-1}\lambda_i u_0^{k-i} - \sum_{i=0}^{k-1}\lambda_i u_0^0\right] - \frac{2K_\gamma}{h^2}[u_1^k - u_0^k - h\phi(t_k)] - g_0^k \\ &= \tau^{-\gamma}\left[u_0^k + \sum_{i=1}^{k-1}\lambda_i u_0^{k-i} - \sum_{i=0}^{k-1}\lambda_i u_0^0\right] - {}_0D_t^\gamma[u_1^k - u_0^0] \\ &\quad + K_\gamma\left[\frac{\partial^2 u_0^k}{\partial x^2} - \frac{2}{h^2}\left\{\frac{h^2}{2}\frac{\partial^2 u_0^k}{\partial t} + 0(h^3)\right\}\right] \\ &= o(\tau + h), \quad 1 \le k \le n. \end{aligned} \tag{18}$$

Similarly, using the Taylor series expansion of u_{m-1}^k about the point (x_m, t_k), we get

$$R_m^k = o(\tau + h), \quad 1 \le k \le n. \tag{19}$$

5.2 Stability

We introduce some relevant notations and properties. Suppose that $u^k = \{u_j^k | 0 \le j \le m, 0 \le k \le n\}$ and $v^k = \{v_j^k | 0 \le j \le m, 0 \le k \le n\}$ are two grid functions on $\Omega_h \times \Omega_\tau$. We introduce the following notations:

$$(u_j^k)_x = \frac{u_{j+1}^k - u_j^k}{h}, \quad (u_j^k)_{\bar{x}} = \frac{u_j^k - u_{j-1}^k}{h}, \quad (u_j^k)_{x\bar{x}} = \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h},$$

$$\langle u_k, v_k\rangle = h\left[\frac{u_0^k v_0^k}{2} + \sum_{j=1}^{m-1} u_j^k v_j^k + \frac{u_m^k v_m^k}{2}\right], \quad \|u^k\| = \langle u^k, u^k\rangle^{1/2}.$$

Let $\tilde{U}_j^k$ be the approximate solution of (15) or (16) and $\epsilon_j^k = U_j^k - \tilde{U}_j^k$, $0 \le j \le m$, $0 \le k \le n$, denote the corresponding error.

Lemma 5.1. *[3] The coefficients λ_i $(i = 0, 1, 2, \dots)$ satisfy*

1. $\lambda_0 = 1$, $\lambda_1 = -\gamma$, $\lambda_i < 0$, $i = 1, 2, \dots$.
2. $\sum_{i=0}^{\infty}\lambda_i = 0$.
3. $\sum_{i=0}^{k-1}\lambda_i > 0$ *and consequently* $-\sum_{i=1}^{k}\lambda_i < 1$, *for all* $k \ge 1$.

Lemma 5.2. *Suppose that ϵ^k $(0 \le k \le n)$ is error of IDAS (15) or (16). Then we have*

$$\|\epsilon^k\|^2 \le -\sum_{i=1}^{k-1}\lambda_i\|\epsilon^{k-i}\|^2 + \sum_{i=0}^{k-1}\lambda_i\|\epsilon^0\|^2, \quad k = 1, 2, \dots n.$$

Proof. For the IDAS defined by (15) or (16), for $1 \le k \le n$, its error satisfies

$$(1 + 2\mu)\epsilon_0^k = -\sum_{i=1}^{k-1}\lambda_i\epsilon_0^{k-i} + \sum_{i=0}^{k-1}\lambda_i\epsilon_0^0 + 2\mu\epsilon_1^k, \tag{20}$$