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Moduli Spaces

Edited by
Leticia Brambila-Paz, Oscar García-Prada,
Peter Newstead and Richard P. Thomas

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Preface

A programme on Moduli Spaces was held from 4 January to 1 July 2011 at the Isaac Newton Institute for Mathematical Sciences in Cambridge, UK. This volume is based on courses and lectures that took place during this semester and reflects some of the main themes that were covered during the activities.

Moduli spaces play a fundamental role in geometry. They are geometric versions of parameter spaces. That is, they are geometric spaces which parametrise something – each point represents one of the objects being parametrised, such as the solution of a particular equation, or a geometric structure on some other object. In the language of physics, a moduli space models the degrees of freedom of the solutions of some system of equations.

The programme was very successful, with a great deal of activity taking place. There were three main areas of research involved in the programme, namely derived categories, Higgs bundles and character varieties, and vector bundles and coherent systems. Topics that were covered included BPS invariants of 3-folds from derived categories of sheaves, and their motivic and categorified refinements, Hodge polynomials of character varieties, motives of moduli spaces of Higgs bundles and their relation to BPS invariants, Gromov–Witten invariants, notions of stability, Bridgeland stability, stability for pairs, geometric invariant theory constructions, wall-crossing formulae using Kirwan blow-ups, d-manifolds, a motivic version of Göttsche's conjecture, the Hilbert scheme of the moduli space of vector bundles, derived categories of quiver representations, mirror symmetry conjecture, ramified non-abelian Hodge theory correspondence, Hitchin fibration and real forms, parahoric bundles, parabolic Higgs bundles and representations of fundamental groups of punctured surfaces, Higgs bundles on Klein surfaces, Higgs bundles and groups of Hermitian type, Higgs bundles over elliptic curves, geometry of moduli spaces of vortices, coherent systems and geometry of moduli of curves, Brill–Noether loci for fixed determinant, Green's conjecture, Butler's conjecture, etc.

We saw progress on many of these topics in real time; it is fair to say that the state of the art looked very different at the end of the six months than it did in the introductory school at the beginning.

Acknowledgements

We are indebted to the authors of these articles for their outstanding contributions and to the referees for the care with which they have read the articles and the helpful suggestions they have made. We thank the speakers in the School on Moduli Spaces held in January 2011 on which this book is based and all participants in the school and in the more extensive programme on Moduli Spaces held from 4 January to 1 July 2011.

Our most grateful thanks are due to the Isaac Newton Institute for funding and hosting this activity. The staff of the Institute were unfailingly helpful. We also acknowledge Cambridge University Press for their help in publishing the volume.

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Contents

1

Introduction to algebraic stacks

K. Behrend

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Abstract

These are lecture notes based on a short course on stacks given at the Isaac Newton Institute in Cambridge in January 2011. They form a self-contained introduction to some of the basic ideas of stack theory.

Contents

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Introduction

Stacks and algebraic stacks were invented by the Grothendieck school of algebraic geometry in the 1960s. One purpose (see [11]) was to give geometric meaning to higher cohomology classes. The other (see [9] and [2]) was to develop a more general framework for studying moduli problems. It is the latter aspect that interests us in this chapter. Since the 1980s, stacks have become an increasingly important tool in geometry, topology and theoretical physics.

Stack theory examines how mathematical objects can vary in families. For our examples, the mathematical objects will be the triangles, familiar from Euclidean geometry, and closely related concepts. At least to begin with, we will let these vary in continuous families, parametrized by topological spaces.

A surprising number of stacky phenomena can be seen in such simple cases. (In fact, one of the founders of the theory of algebraic stacks, M. Artin, is famously reputed to have said that one need only understand the stack of triangles to understand stacks.)

This chapter is divided into three parts, Sections 1.1, 1.2, and 1.3. Section 1.1 is a very leisurely and elementary introduction to stacks, introducing the main ideas by considering a few elementary examples of topological stacks. The only prerequisites for this section are basic undergraduate courses in abstract algebra (groups and group actions) and topology (topological spaces, covering spaces, the fundamental group).

Section 1.2 introduces the basic formalism of stacks. The prerequisites are the same, although this section is more demanding than the preceding one.

Section 1.3 introduces algebraic stacks, culminating in the Riemann–Roch theorem for stacky curves. The prerequisite here is some basic scheme theory.

We do not cover much of the "algebraic geometry" of algebraic stacks, but we hope that these notes will prepare the reader for the study of more advanced texts, such as [16] or the forthcoming book.[1]

The following outline uses terminology that will be explained in the body of the text.

The first fundamental notion is that of a *symmetry groupoid of a family of objects*. This is introduced first for discrete and then for continuous families of triangles.

In Sections 1.1.1–1.1.3, we consider Euclidean triangles up to similarity (the stack of such triangles is called $\mathfrak{M}$). We define what a fine moduli space is, and show how the symmetries of the isosceles triangles and the equilateral triangle prevent a fine moduli space from existing. We study the coarse moduli space of triangles, and discover that it parametrizes a *modular family*, even though this family is, of course, not universal.

Sections 1.1.4–1.1.6, introduce other examples of moduli problems. In Section 1.1.4, we encounter a fine moduli space (the fine moduli space of scalene triangles); in Section 1.1.5, where we restrict attention to isosceles triangles, we encounter a coarse moduli space supporting several non-isomorphic modular families. Restricting attention entirely to the equilateral triangle, in Section 1.1.6, we come across a coarse moduli space that parametrizes a modular family which is versal, but not universal.

In Section 1.1.7, we finally exhibit an example of a coarse moduli space which does not admit any modular family at all. We start studying *oriented triangles*. We will eventually prefer working with oriented triangles, because they are more closely related to algebraic geometry. The stack of oriented triangles is called $\widetilde{\mathfrak{M}}$.

In Section 1.1.8, we first make a few general and informal remarks about stacks and their role in the study of moduli problems.

The second fundamental concept is that of *versal family*. Versal families replace universal families, where the latter do not exist. Stacks that admit versal families are called *geometric*, which means *topological* in Sections 1.1 and 1.2, but will mean *algebraic* in Section 1.3.

We introduce versal families in Section 1.1.9, and give several examples. We explain how a stack which admits a versal family is essentially equal to the stack of 'generalized moduli maps' (or torsors, in more advanced terminology).

In Section 1.1.10, we start including degenerate triangles in our examinations: triangles whose three vertices are collinear. The main reason we do this

[1] Contact Martin Olsson, www.math.berkeley.edu/~molsson.