Jack Hale

Applied Mathematical Sciences 3

Theory of Functional Differential Equations

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Preface

Since the publication of my lecture notes, *Functional Differential Equations* in the Applied Mathematical Sciences series, many new developments have occurred. As a consequence, it was decided not to make a few corrections and additions for a second edition of those notes, but to present a more comprehensive theory. The present work attempts to consolidate those elements of the theory which have stabilized and also to include recent directions of research.

The following chapters were not discussed in my original notes. Chapter 1 is an elementary presentation of linear differential difference equations with constant coefficients of retarded and neutral type. Chapter 4 develops the recent theory of dissipative systems. Chapter 9 is a new chapter on perturbed systems. Chapter 11 is a new presentation incorporating recent results on the existence of periodic solutions of autonomous equations. Chapter 12 is devoted entirely to neutral equations. Chapter 13 gives an introduction to the global and generic theory. There is also an appendix on the location of the zeros of characteristic polynomials.

The remainder of the material has been completely revised and updated with the most significant changes occurring in Chapter 3 on the properties of solutions, Chapter 5 on stability, and Chapter 10 on behavior near a periodic orbit.

It is impossible to thank individually by name all my friends, colleagues, and students who have helped me over the years to understand something about functional differential equations. Each of these persons will recognize their influence on the presentation. However, Chapter 13 on the global theory could not have been written without the special assistance given by John Mallet-Paret and Waldyr Oliva. To Pedro Martinez-Amores, I also owe a special thanks for his reading of the original manuscript. His criticisms and constructive comments led to a much better presentation. In the final preparation of the manuscript, Sandra Spinacci went far beyond the call of duty, working incessantly and under very strained conditions. I also thank Eleanor Addison for her work on the drawings. For a careful reading of the galley proofs, I am indebted to P. Lima, H. Rodrigues, and P. Táboas. Finally, it was a pleasure to work with the professional staff of Springer-Verlag, especially Elizabeth Kaplan.

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Contents

Introduction

In many applications, one assumes the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. If it is also assumed that the system is governed by an equation involving the state and rate of change of the state, then, generally, one is considering either ordinary or partial differential equations. However, under closer scrutiny, it becomes apparent that the principle of causality is often only a first approximation to the true situation and that a more realistic model would include some of the past states of the system. Also, in some problems it is meaningless not to have dependence on the past. This has been known for some time, but the theory for such systems has been extensively developed only recently. In fact, until the time of Volterra [1] most of the results obtained during the previous 150 years were concerned with special properties for very special equations. There were some very interesting developments concerning the closure of the set of exponential solutions of linear equations and the expansion of solutions in terms of these special solutions. On the other hand, there seemed to be little concern about a qualitative theory in the same spirit as for ordinary differential equations.

In his research on predator–prey models and viscoelasticity, Volterra [1, 2] formulated some rather general differential equations incorporating the past states of the system. Also, because of the close connection between the equations and specific physical systems, Volterra attempted to introduce a concept of energy function for these models. He then exploited the behavior of this energy function to study the asymptotic behavior of the system in the distant future. These beautiful papers were almost completely ignored by other workers in the field and therefore did not have much immediate impact.

In the late thirties and early forties, Minorsky [1], in his study of ship stabilization and automatic steering, pointed out very clearly the importance of the consideration of the delay in the feedback mechanism. The great interest in control theory during these and later years has certainly contributed significantly to the rapid development of the theory of differential equations with dependence on the past state.

In the late forties and early fifties, a few books appeared which presented the current status of the subject and certainly greatly influenced later developments. In his book, Mishkis [1] introduced a general class of equations with delayed arguments and laid the foundation for a general theory of linear systems. In their monograph at the Rand Corporation, Bellman and Danskin [1] pointed out the diverse applications of equations containing past information to other areas such as biology and economics. They also presented a well organized theory of linear equations with constant coefficients and the beginnings of stability theory. A more extensive development of these ideas is in the book of Bellman and Cooke [1]. In his book on stability theory, Krasovskii [1] presented the theory of Liapunov functionals emphasizing the important fact that some problems in such systems are more meaningful and amenable to solution if one considers the motion in a function space even though the state variable is a finite-dimensional vector.

With such clear indications of the importance of these systems in the applications and also with the number of interesting mathematical problems involved, it is not surprising that the subject has undergone a rapid development in the last twenty five years. New applications also continue to arise and require modifications of even the definition of the basic equations. We list below a few types of equations that have been encountered merely to give an idea of the diversity and give appropriate references for the specific application.

The simplest type of past dependence in a differential equation is that in which the past dependence is through the state variable and not the derivative of the state variable, the so-called retarded functional differential equations or retarded differential difference equations. For a discussion of the physical applications of the differential difference equation

$$\dot{x}(t) = F(t, x(t), x(t - r)) \qquad \dot{x} = \frac{dx}{dt}$$

to control problems, see Minorsky [2, Ch. 21]. Lord Cherwell (see Wright [1, 2]) has encountered the differential difference equation

$$\dot{x}(t) = -\alpha x(t - 1)[1 + x(t)]$$

in his study of the distribution of primes. Variants of this equation have also been used as models in the theory of growth of a single species (see

Cunningham [1]). Dunkel [1] suggested the more general equation

$$\dot{x}(t) = -\alpha\left[\int_{-1}^{0} x(t+\theta)d\eta(\theta)\right][1 + x(t)]$$

for the growth of a single species.

In his study of predator–prey models, Volterra [1] had earlier investigated the equations

$$\dot{x}(t) = \left[\varepsilon_1 - \gamma_1 y(t) - \int_{-r}^{0} F_1(\theta)y(t+\theta)d\theta\right]x(t)$$

$$\dot{y}(t) = \left[-\varepsilon_1 + \gamma_2 x(t) + \int_{-r}^{0} F_2(\theta)x(t+\theta)d\theta\right]y(t)$$

where x and y are the number of prey and predators, respectively, and all constants and functions are non-negative. For similar models, Wangersky and Cunningham [1, 2], have also used the equations

$$\dot{x}(t) = \alpha x(t)\left[\frac{m - x(t)}{m}\right] - bx(t)y(t)$$

$$\dot{y}(t) = -\beta y(t) + cx(t-r)y(t-r)$$

for predator–prey models.

In an attempt to explain the circummutation of plants (and especially the sunflower), Israelson and Johnsson [1, 2] have used the equation

$$\dot{\alpha}(t) = -k\int_{1}^{\infty} f(\theta)\sin\alpha(t - \theta - t_0)d\theta$$

as a model, where α is the angle the top of the plant makes with the vertical (see also Klein [1]). For other applications, see Johnson and Karlsson [1].

Under suitable assumptions, the equation

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - T_i)$$

is a suitable model for describing the mixing of a dye from a central tank as dyed water circulates through a number of pipes. An application to the distribution in man of labeled albumin as it circulates from the blood stream through the interstitial fluids and back to the blood stream is discussed by Bailey and Reeve [1] (see also Bailey and Williams [1]). Boffi and Scozzafava [1, 2] have also encountered this equation in transport problems.

In an attempt to describe the spread of measles in a metropolitan area, London and Yorke [1] have encountered the equation

$$\dot{S}(t) = -\beta(t)S(t)[2\gamma + S(t-14) - S(t-12)] + \gamma$$

where $S(t)$ is the number of susceptible individuals at time t, γ is the rate at which individuals enter the population, $\beta(t)$ is a function characteristic of the population, and an individual exposed at time t is infectious in the time interval $[t - 14, t - 12]$.

In an analysis of gonorrhea, Cooke and Yorke [1] have studied the equation

$$\dot{I}(t) = g(I(t - L_1)) - g(I(t - L_2))$$

where I represents the number of infectious individuals and g is a nonnegative function vanishing outside a compact interval.

A more general equation describing the spread of disease taking into account age dependence was given by Cooke [1] and Hoppenstadt and Waltman [1]. For other equations that occur in the theory of epidemics, see Waltman [1]. For other models in the biomedical sciences, see Banks [1]. Grossberg [1, 2] has encountered interesting differential equations in the theory of learning.

The equation

$$\dot{x}(t) = -\int_{t-r}^{t} a(t - u)g(x(u))du$$

was encountered by Ergen [1] in the theory of a circulating fuel nuclear reactor and has been extensively studied by Levin and Nohel [1]. In this model, x is the neutron density. It is also a good model in one-dimensional viscoelasticity in which x is the strain and a is the relaxation function.

Taking into account the transmission time in the triode oscillator, Rubanik [1, p. 130] has encountered the van der Pol equation

$$\ddot{x}(t) + \alpha\dot{x}(t) - f(x(t - r))\dot{x}(t - r) + x(t) = 0$$

with the delayed argument r. Taking into account the retarded connections between oscillating systems, Starik [1] has encountered the system

$$\begin{aligned} \ddot{y}(t) + [\omega^2 + \varepsilon\lambda \sin \phi(t - r_1)]y(t) &= -\varepsilon[h\dot{y}(t) + \gamma y^3(t - r_2)] \\ I\ddot{\phi}(t) &= \varepsilon[L(\dot{\phi}(t)) - H\phi(t) - {\sigma_1}^2 y^2(t - r_3)\cos \phi(t) \\ &\quad - \sigma_2 \sin \phi(t) - \sigma_3 \cos \phi(t)]. \end{aligned}$$

In the theory of optimal control, Krasovskii [2] has studied extensively the system

$$\dot{x}(t) = P(t)x(t) + B(t)u(t)$$

$$y(t) = Q(t)x(t)$$

$$\dot{u}(t) = \int_{-r}^{0} [d_\theta \eta(t, \theta)]y(t + \theta) + \int_{-r}^{0} [d_\theta \mu(t, \theta)]u(t + \theta).$$

There are also a number of applications in which the delayed argument occurs in the derivative of the state variable as well as in the independent variable, the so-called neutral differential difference equations. Such problems are more difficult to motivate but often arise in the study of two or more

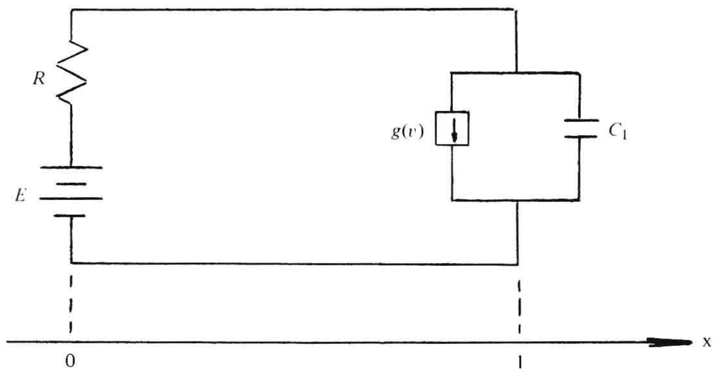

Figure I.1

simple oscillatory systems with some interconnections between them. For simplicity, it is usually assumed that the interaction of the components of the coupled systems takes place immediately. In many cases, the time for the interaction to take place is important even in determining the qualitative behavior of the system. It often occurs that the connection between the coupled systems can be adequately described by a system of linear hyperbolic partial differential equations with the motion of each individual system being described by a boundary condition. In some cases, the connection through the partial differential equations (considered as a connection by a traveling wave) can be replaced by connections with delays. Generally, the resulting ordinary differential equations involve delays in the highest order derivatives. A general discussion of when this process is valid may be found in Rubanik [1] and Cooke and Krumme [1].

For example, Brayton [1] considered the lossless transmission line connected as shown in Figure I.1, where $g(v)$ is a nonlinear function of v and gives the current in the indicated box in the direction shown. This problem may be described by the following system of partial differential equations

$$L\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}, \qquad C\frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}, \qquad 0 < x < 1, t > 0,$$

with the boundary conditions

$$E - v(0, t) - Ri(0, t) = 0, \qquad C_1\frac{dv(1, t)}{dt} = i(1, t) - g(v(1, t)).$$

We now indicate how one can transform this problem into a differential equation with delays. If $s = (LC)^{-1/2}$ and $z = (L/C)^{1/2}$, then the general solution of the partial differential equation is

$$v(x, t) = \phi(x - st) + \psi(x + st)$$

$$i(x, t) = \frac{1}{z}[\phi(x - st) - \psi(x + st)]$$

or

$$2\phi(x - st) = v(x, t) + zi(x, t)$$

$$2\psi(x + st) = v(x, t) - zi(x, t).$$

This implies

$$2\phi(-st) = v\left(1, t + \frac{1}{s}\right) + zi\left(1, t + \frac{1}{s}\right)$$

$$2\psi(st) = v\left(1, t - \frac{1}{s}\right) - zi\left(1, t - \frac{1}{s}\right)$$

Using these expressions in the general solution and using the first boundary condition at $t - (1/s)$, one obtains

$$i(1, t) - Ki\left(1, t - \frac{2}{s}\right) = \alpha - \frac{1}{z}v(1, t) - \frac{K}{z}v\left(1, t - \frac{2}{s}\right)$$

where $K = (z - R)/(z + R)$, $\alpha = 2E/(z + R)$. Inserting the second boundary condition and letting $u(t) = v(1, t)$, we obtain the equation

$$\dot{u}(t) - K\dot{u}\left(t - \frac{2}{s}\right) = f\left(u(t), u\left(t - \frac{2}{s}\right)\right)$$

where $s = \sqrt{LC}$,

$$C_1 f(u(t), u(t - r)) = \alpha - \frac{1}{z}u(t) - \frac{K}{z}u(t - r) - g(u(t)) + K\,g(u(t - r)),$$

all constants are positive and depend on the parameters in the original equations. Also, if $R > 0$, then $K < 1$.

If generalized solutions of the original partial differential equation were considered, the delay equation would require differentiating the difference $u(t) - Ku(t - (2/s))$ rather than each term separately; that is, one would consider the equation

$$\frac{d}{dt}\left[u(t) - Ku\left(t - \frac{2}{s}\right)\right] = f\left(u(t), u\left(t - \frac{2}{s}\right)\right).$$

The prescription for passing from a linear partial differential equation with nonlinear boundary conditions to a delay equation is certainly not unique and other transformations may be desirable in certain situations. This fact is illustrated following the ideas of Lopes [1]. Let $|K| < 1$ (i.e., $R > 0$) and let p be any solution of the difference equation

$$p(t) - Kp\left(t - \frac{2}{s}\right) = -b(t),\ b(t) = zE\left(t - \frac{1}{s}\right)\Big/(z + R)$$

If E is periodic, one can choose p periodic with the same period. Using the first boundary condition at $t - (1/s)$ and the general solution, one obtains

$$\phi(1 - st) = b(t) - K\psi(st - 1).$$

If $w(t) = \psi(1 + st) - p(t)$, then evaluation in the general solution gives

$$v(1, t) = w(t) - Kw(t - r)$$

$$i(1, t) = -\frac{1}{z} w(t) - \frac{K}{z} w(t - r) + q$$

where $r = 2/s$, $zq(t) = -p(t) - Kp(t - r) + b(t)$. Using the second boundary condition one obtains the equation

$$C_1 \frac{d}{dt}[w(t) - Kw(t - r)] = q - \frac{1}{z} w(t) - \frac{K}{z} w(t - r) - g(w(t) - Kw(t - r))$$

In his consideration of shunted transmission lines, Lopes [2] encountered equations of the above type with two delays.

Sometimes boundary control of a linear hyperbolic equation can be more effectively studied by investigating the corresponding control problem for the above transformed equations (see Banks and Kent [1]).

Another similar equation encountered by Rubanik [1] in his study of vibrating masses attached to an elastic bar is

$$\ddot{x}(t) + \omega_1{}^2 x(t) = \varepsilon f_1(x(t), \dot{x}(t), y(t), \dot{y}(t)) + \gamma_1 \ddot{y}(t - r)$$

$$\ddot{y}(t) + \omega_2{}^2 y(t) = \varepsilon f_2(x(t), \dot{x}(t), y(t), \dot{y}(t)) + \gamma_2 \ddot{x}(t - r)$$

In studying the collision problem in electrodynamics, Driver [1] considered systems of the type

$$\dot{x}(t) = f_1(t, x(t), x(g(t))) + f_2(t, x(t), x(g(t)))\dot{x}(g(t))$$

where $g(t) \leq t$. In the same problem, one encounters delays g which depend also upon x.

El'sgol'tz [1, 2], Sabbagh [1], and Hughes [1] have considered the variational problem of minimizing

$$V(x) = \int_0^1 F(t, x(t), x(t - r), \dot{x}(t), \dot{x}(t - r))dt$$

over some class of functions x. Generally, the Euler equations are of the form

$$\ddot{x}(t) = f(t, x(t), x(t - r), \dot{x}(t), \dot{x}(t - r), \ddot{x}(t - r))$$

with some appropriate boundary conditions.

In the slowing down of neutrons in a nuclear reactor, the asymptotic behavior as $t \to \infty$ of the equation

$$x(t) = \int_t^{t+1} k(s)x(s)ds$$

or

$$\dot{x}(t) = k(t + 1)x(t + 1) - k(t)x(t)$$