ASYMPTOTIC EXPANSIONS

by

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PREFACE

In 1943, at the request of the Admiralty Computing Service, I wrote a short monograph on *The Asymptotic Expansion of a Function Defined by a Definite Integral or Contour Integral*. This was one of a series of monographs intended for use in Admiralty Research Establishments, on topics which appeared to be inadequately covered in easily accessible literature. It evidently met a need of the time, since a revised edition was issued in 1946 and had a wide circulation.

The Admiralty monograph has long been unobtainable, and several of my friends have urged me to write this more extensive book on the same general lines. There are few theorems; the aim is the modest one of explaining the methods which are available, and illustrating them by means of a few of the more important special functions.

I must express my thanks to Professor Arthur Erdélyi for the generous advice and encouragement he has given me during the writing of this book.

E. T. C.

CONTENTS

Chapter 1

INTRODUCTION

Although the subject of 'Modern Analysis' had its beginnings in the seventeenth century, the mathematicians of the eighteenth century were often more interested in the formal use of infinite processes than in their rigorous proofs. Some of these formal results are very striking.

For example, in 1730, Stirling in his *Methodus Differentialis* gave an infinite series for $\log(m!)$ which, in modern notation, would be written as

$$\log(m!) = z\log z - z + \tfrac{1}{2}\log(2\pi) + \sum_{1}^{\infty} \frac{B_{2n}(\frac{1}{2})}{(2n-1)(2n)\,z^{2n-1}},$$

where $z = m + \frac{1}{2}$, and $B_n(x)$ is Bernoulli's polynomial defined by

$$\frac{z\,e^{xz}}{e^z - 1} = \sum_{0}^{\infty} B_n(x)\frac{z^n}{n!}.$$

Stirling gave the first five coefficients and a recurrence formula for successively determining the coefficients. A similar formula

$$\log(m!) = (m+\tfrac{1}{2})\log m - m + \tfrac{1}{2}\log(2\pi) + \sum_{1}^{\infty} \frac{B_{2n}(0)}{(2n-1)(2n)\,m^{2n-1}}$$

was subsequently given by De Moivre. Both these series are divergent; yet Stirling was able to calculate $\log_{10}(1000!)$, a number between 2567 and 2568, to ten places of decimals by taking only the first few terms of his series. Any partial sum of either of these divergent series is an approximation of $\log(m!)$ with an error of the order of the first term omitted; and, since these terms decrease very rapidly initially, the sum of a few terms may give a very good approximation.

Another interesting result, due to Euler, is that

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{m} - \log m = \gamma + \frac{1}{2m} - \sum_{1}^{\infty} \frac{B_{2n}(0)}{(2n)\,m^{2n}},$$

where γ is Euler's constant. Euler pointed out that the series is divergent for $m = 1$, but he used the series with $m = 10$ to calculate γ correct to 15 places of decimals; yet the series diverges for all values of m. The series is an alternating series, whose terms at first decrease in numerical value; the partial sums lie alternately above and below the desired 'sum', and the accuracy of an approximation by a partial sum can never be better than the least term.

In his *Théorie analytique des probabilités*, published in Paris in 1812, Laplace introduced two new ideas. He showed that the error function

$$\operatorname{Erf}(T) = \int_0^T e^{-t^2}\,dt$$

can be represented by a convergent power series

$$\sum_0^\infty (-1)^n \frac{T^{2n+1}}{(2n+1).n!},$$

which he called a *série-limite*, because successive partial sums lie alternately above and below the value of the integral, and so provide upper and lower bounds for its value. But when T is large, the series converges so slowly that these bounds are of little use. He also obtained by integration by parts a *série-limite* for the related function

$$\operatorname{Erfc}(T) = \int_T^\infty e^{-t^2}\,dt,$$

namely,
$$\frac{e^{-T^2}}{2T}\left\{1-\frac{1}{2T^2}+\frac{1.3}{(2T^2)^2}-\frac{1.3.5}{(2T^2)^3}+\dots\right\}.$$

Laplace remarked 'La série a l'inconvénient de finir par être divergente', yet its *série-limite* property made it possible to compute from it values of $\operatorname{Erfc}(T)$ for large values of T.

The second idea was that the value of an integral

$$\int \phi(x)\,\{u(x)\}^s\,dx$$

when s is large, depends on the behaviour of $u(x)$ near its stationary points. Laplace used this to obtain the result

$$s! = s^{s+\frac{1}{2}}\,e^{-s}\,\sqrt{(2\pi)}\left[1+\frac{1}{12s}+\dots\right],$$

which is frequently, but incorrectly, called Stirling's approximation to the factorial.

Legendre, in his *Traité des fonctions elliptiques* (1825–28), called an infinite series *demi-convergente* if it represented a given function in the sense that the error committed by stopping at any term is of the order of the first term omitted. The term was unfortunate as this property has nothing to do with convergence; for example, the series

$$\sum_0^\infty (-1)^n r^n \quad (0<r<1) \qquad \text{and} \qquad \sum_1^\infty \frac{1}{n^2}$$

are both convergent, the first being semiconvergent, the second not semiconvergent. Yet the usage has persisted; it occurs, for instance, in Jahnke and Emde's *Funktionentafeln*. A semiconvergent series is nowadays said to be *asymptotic*.

During the nineteenth century, asymptotic expansions were obtained for many of the special functions of analysis, sometimes only formally, sometimes with a rigorous discussion of the order of magnitude of the error. Of particular interest is Stieltjes's [26] doctorate thesis in which he examined the error committed by stopping at the least term of the asymptotic representations of certain important special functions, and showed how the approximation so obtained could be improved.

The modern theory of asymptotic expansions originated in the work of Poincaré [22]. The subject falls roughly into two parts. The first part deals with the summability of asymptotic series, and with the validity of such operations as term by term differentiation or integration; the second is concerned with the actual construction of a series which represents a given function asymptotically.

This tract discusses the asymptotic representation of a function defined by a definite integral or contour integral, usually an analytic function of a complex variable z. If z is complex, we denote its real and imaginary parts by $\mathscr{R}z$ and $\mathscr{I}z$. If z is written in polar form

$$z = r\cos\theta + ir\sin\theta \quad (r>0),$$

we call θ the phase of z, and write $\theta = \mathrm{ph}\, z$. The phase of z is not unique; it is determined by z only up to an additive multiple of 2π. The principal value of $\mathrm{ph}\, z$ satisfies the inequality

$-\pi < \mathrm{ph}\, z \leqslant \pi$; but since non-integral powers of z will occur, values of $\mathrm{ph}\, z$ other than the principal value turn out to be important.

The terminology and notation used here for the special functions are those of Erdélyi, Magnus, Oberhettinger and Tricomi's three-volume work *Higher Transcendental Functions* (McGraw-Hill, 1953). As this book is very well indexed, no references to it will be given unless it is absolutely necessary. It should be pointed out that the notation of this book is not always the same as that of Whittaker and Watson's *Modern Analysis* (Cambridge, 1920).

The bibliography on pages 118 and 119 is merely a list of works to which reference is made in the text. For example, Poincaré [22] on page 3 refers to Henri Poincaré's *Acta Mathematica* paper which is the twenty-second item in the bibliography.

CHAPTER 2

PRELIMINARIES

1. Asymptotic sequences

Let $f(z)$ and $\phi(z)$ be two functions defined on a set R in the complex plane, and let z_0 be a limit point of R, possibly the point at infinity. For example, R may be a sector

$$0 < |z| < \infty, \quad \alpha < \operatorname{ph} z < \beta,$$

and z_0 might then be the origin or the point at infinity. By a neighbourhood of z_0 (more strictly a spherical neighbourhood), we mean an open disc $|z-z_0| < \delta$ if z_0 is at a finite distance, a region $|z| > \delta$ if z_0 is the point at infinity.

In the usual notation we write $f = O(\phi)$ if there exists a constant A such that $|f| \leqslant A\,|\phi|$ for all z in R. We also write $f = O(\phi)$ as $z \to z_0$ if there exists a constant A and a neighbourhood U of z_0 such that $|f| \leqslant A\,|\phi|$ for all points in the intersection of U and R; and $f = o(\phi)$ as $z \to z_0$ if, for any positive number ϵ, there exists a neighbourhood U of z_0 such that $|f| \leqslant \epsilon\,|\phi|$ for all points z of the intersection of U and R. More simply, if ϕ does not vanish on R, $f = O(\phi)$ means that f/ϕ is bounded, $f = o(\phi)$ that f/ϕ tends to zero as $z \to z_0$.

A sequence of functions $\{\phi_n(z)\}$ is called an *asymptotic sequence* as $z \to z_0$ if there is a neighbourhood of z_0 in which none of the functions vanish (excepting the point z_0) and if for all n

$$\phi_{n+1} = o(\phi_n) \quad \text{as} \quad z \to z_0.$$

For example, if z_0 is finite, $\{(z-z_0)^n\}$ is an asymptotic sequence as $z \to z_0$; and $\{z^{-n}\}$ is as $z \to \infty$.

2. Poincaré's definition of an asymptotic expansion

The formal series

$$\sum_0^\infty a_n\,\phi_n(z),$$

not necessarily convergent, is said to be an asymptotic expansion of $f(z)$ in Poincaré's sense, with respect to the asymptotic sequence $\{\phi_n(z)\}$ if, for every value of m,

$$f(z) - \sum_0^m a_n \phi_n(z) = o(\phi_m(z)),$$

as $z \to z_0$. Since

$$f(z) - \sum_0^{m-1} a_n \phi_n(z) = a_m \phi_m(z) + o(\phi_m(z)),$$

the partial sum
$$\sum_0^{m-1} a_n \phi_n(z)$$

is an approximation to $f(z)$ with an error $O(\phi_m)$ as $z \to z_0$; this error is of the same order of magnitude as the first term omitted. If such an asymptotic expansion exists, it is unique, and the coefficients are given successively by

$$a_m = \lim_{z \to z_0} \left\{ f(z) - \sum_0^{m-1} a_n \phi_n(z) \right\} \Big/ \phi_m(z).$$

If a function possesses an asymptotic expansion in this sense, we write

$$f(z) \sim \sum_0^\infty a_n \phi_n(z).$$

A partial sum of this formal series will often be called an *asymptotic approximation* to $f(z)$. The first term is called the *dominant term*; and we frequently write $f(z) \sim a_0 \phi_0(z)$, meaning that $f(z)/\phi_0(z)$ tends to a_0 as $z \to z_0$.

The definition has been given for functions of a complex variable z, but it can easily be modified for functions of a real variable x. If the limit point x_0 is finite, R could be an open interval of which x_0 is an internal or end-point; and a neighbourhood of x_0 is an open interval $|x - x_0| < \delta$. But if x_0 is infinite, we have to discriminate between $x \to +\infty$, in which case R could be a semi-infinite interval $x > a$ say, and $x \to -\infty$, in which case R could be, say, $x < b$. There are cases when R is a discrete set; for instance, it might be necessary to find an asymptotic expansion for the nth partial sum of an infinite series when n is large, but such problems lie, in the main, outside the scope of this tract.

The form of an asymptotic expansion evidently depends on the choice of the asymptotic sequence. For example, as $z \to \infty$,

$$\frac{1}{z-1} \sim \sum_{1}^{\infty} \frac{1}{z^n},$$

and

$$\frac{1}{z-1} \sim \sum_{1}^{\infty} \frac{z+1}{z^{2n}}.$$

In these examples it happens that the asymptotic expansions are convergent series.

Again, two functions may have the same asymptotic expansion. For example, if $-\frac{1}{2}\pi+\delta \leqslant \operatorname{ph} z \leqslant \frac{1}{2}\pi-\delta$, where $0 < \delta < \frac{1}{2}\pi$, the two functions

$$\frac{1}{z+1}, \quad \frac{1}{z+1}+e^{-z},$$

both have the asymptotic expansion

$$\sum_{1}^{\infty} \frac{(-1)^{n-1}}{z^n}$$

as $z \to \infty$, since $z^n e^{-z}$ tends to zero as $z \to \infty$ in the given sector.

3. Asymptotic power series

If the limit point z_0 is at a finite distance, it is transformed into the point at infinity by $z' = 1/(z-z_0)$. We shall suppose that this has been done and consider only asymptotic expansions as $z \to \infty$ in a sector $\alpha < \operatorname{ph} z < \beta$; or, in the case of a function of a real variable x, as $x \to +\infty$ or as $x \to -\infty$.

The simplest type of asymptotic sequence as $z \to \infty$ is $\{\phi(z)/z^n\}$. If a function $f(z)$ possesses an asymptotic expansion with respect to this sequence, say

$$f(z) \sim \phi(z) \sum_{0}^{\infty} \frac{a_n}{z^n},$$

this implies that

$$\frac{f(z)}{\phi(z)} \sim \sum_{0}^{\infty} \frac{a_n}{z^n},$$

the latter series being an asymptotic expansion with respect to the sequence $\{1/z^n\}$. An asymptotic expansion with respect to the sequence $\{1/z^n\}$ is called an *asymptotic power series*.

4. Calculations with asymptotic power series

Asymptotic power series and convergent power series possess very similar formal properties. The main results are stated below, first for the case of a real variable. It is assumed that $f(x)$ and $g(x)$ possess asymptotic expansions

$$f(x) \sim \sum_0^\infty \frac{a_n}{x^n}, \quad g(x) \sim \sum_0^\infty \frac{b_n}{x^n}$$

as $x \to +\infty$.

(i) *If A is a constant,*

$$Af(x) \sim \sum_0^\infty \frac{Aa_n}{x^n},$$

(ii)
$$f(x)+g(x) \sim \sum_0^\infty \frac{a_n+b_n}{x^n}.$$

These results follow at once from the definition.

(iii)
$$f(x)g(x) \sim \sum_0^\infty \frac{c_n}{x^n},$$

where $$c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_{n-1} b_1 + a_n b_0.$$

For any positive integer N,

$$f(x) = a_0 + \frac{a_1}{x} + \ldots + \frac{a_N}{x^N} + O\left(\frac{1}{x^{N+1}}\right),$$

$$g(x) = b_0 + \frac{b_1}{x} + \ldots + \frac{b_N}{x^N} + O\left(\frac{1}{x^{N+1}}\right),$$

and hence $$f(x)g(x) = c_0 + \frac{c_1}{x} + \ldots + \frac{c_N}{x^N} + O\left(\frac{1}{x^{N+1}}\right),$$

which was to be proved.

It follows that any positive integral power of $f(x)$ possesses an asymptotic power series expansion, and so also does any polynomial in $f(x)$.

(iv) *If $a_0 \neq 0$, then*

$$\frac{1}{f(x)} \sim \frac{1}{a_0} + \sum_1^\infty \frac{d_n}{x^n},$$

as $x \to +\infty$.

In the first place $1/f(x)$ tends to a finite limit $1/a_0$ as $x \to \infty$. Next

$$\left\{\frac{1}{f(x)} - \frac{1}{a_0}\right\} \bigg/ \frac{1}{x} = x\left\{\frac{1}{a_0 + (a_1/x) + O(1/x^2)} - \frac{1}{a_0}\right\}$$

$$= \frac{-a_1 + O(1/x)}{a_0\{a_0 + (a_1/x) + O(1/x^2)\}} \to -\frac{a_1}{a_0^2}.$$

Similarly,
$$\left\{\frac{1}{f(x)} - \frac{1}{a_0} + \frac{a_1}{a_0^2 x}\right\} \bigg/ \frac{1}{x^2} \to \frac{a_1^2 - a_0 a_2}{a_0^3},$$

and so on. The successive coefficients d_n are found in a similar way.

More generally, any rational function of $f(x)$ has an asymptotic power series expansion provided that the denominator does not tend to zero as $x \to +\infty$.

(v) *If $f(x)$ is continuous when $x > a > 0$, then, if $x > a$,*

$$F(x) = \int_x^\infty \left\{f(t) - a_0 - \frac{a_1}{t}\right\} dt$$

has the asymptotic power series expansion

$$F(x) \sim \frac{a_2}{x} + \frac{a_3}{2x^2} + \dots + \frac{a_{n+1}}{nx^n} + \dots$$

as $x \to +\infty$.

Since $f(t) - a_0 - a_1/t$ is continuous when $t > a$ and is $O(1/t^2)$ as $t \to +\infty$, the integral $F(x)$ exists for $x > a$.

Since
$$F(x) = \int_x^\infty \left\{\sum_2^m \frac{a_n}{t^n} + O\left(\frac{1}{t^{m+1}}\right)\right\} dt$$

for every integer $m \geqslant 2$, we have

$$F(x) = \sum_2^m \frac{a_n}{(n-1)\,x^{n-1}} + O\left(\frac{1}{x^m}\right)$$

$$= \sum_1^{m-1} \frac{a_{n+1}}{nx^n} + O\left(\frac{1}{x^m}\right)$$

as $x \to +\infty$; and the result follows.

(vi) *If $f(x)$ has a continuous derivative $f'(x)$, and if $f'(x)$ possesses an asymptotic power series expansion as $x \to +\infty$, the latter expansion is*

$$f'(x) \sim -\sum_2^\infty \frac{(n-1)\,a_{n-1}}{x^n}.$$

For suppose that
$$f'(x) \sim \sum_0^\infty \frac{b_n}{x^n},$$
as $x \to \infty$. Now, since $f'(x)$ is continuous,
$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)\,dt \\ &= b_0(y-x) + b_1 \log\frac{y}{x} + \int_x^y \left\{f'(t) - b_0 - \frac{b_1}{t}\right\} dt. \end{aligned}$$
But $f(y) \to a_0$ as $y \to +\infty$, and
$$\int_x^\infty \left\{f'(t) - b_0 - \frac{b_1}{t}\right\} dt$$
is convergent since the integrand is $O(1/t^2)$. It follows that $b_0 = b_1 = 0$ and that
$$a_0 - f(x) = \int_x^\infty \left\{f'(t) - b_0 - \frac{b_1}{t}\right\} dt.$$

By (v),
$$a_0 - f(x) \sim \sum_1^\infty \frac{b_{n+1}}{nx^n} \quad \text{as} \quad x \to +\infty.$$
But we know that
$$a_0 - f(x) \sim -\sum_1^\infty \frac{a_n}{x^n}.$$

Since an asymptotic power-series expansion is unique,
$$b_{n+1} = -na_n,$$
that is
$$f'(x) \sim -\sum_2^\infty \frac{(n-1)\,a_{n-1}}{x^n},$$
as $x \to +\infty$. In other words, the asymptotic expansion is obtained by formal term by term differentiation.

These results have been stated for functions of a real variable x as $x \to +\infty$. They could have been stated, almost word for word, for functions of a complex variable z as $z \to \infty$ either in a sector or in a whole neighbourhood of the point at infinity.

(vi) can be modified in the case of analytic functions of a complex variable z which are, by definition, differentiable. The result in this case is:

(vii) *If $f(z)$ is an analytic function, regular in the region R defined by $|z| > a$, $\alpha < |\mathrm{ph}\, z| < \beta$, and if*
$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

uniformly in ph z *as* $|z| \to \infty$ *in any closed sector contained in* R, *then*

$$f'(z) \sim -\frac{a_1}{z^2} - \frac{2a_2}{z^3} - \frac{3a_3}{z^4} - \dots$$

uniformly in ph z *as* $|z| \to \infty$ *in any closed sector contained in* R.

When we say that the asymptotic power series expansion of $f(z)$ holds uniformly in ph z as $|z| \to \infty$ in a closed sector

$$\alpha_1 \leqslant \text{ph}\, z \leqslant \beta_1$$

contained in R, we mean that, for every integer m,

$$f(z) = \sum_{n=0}^{m-1} \frac{a_n}{z^n} + \frac{\phi_m(z)}{z^m},$$

where $\phi_m(z)$ is bounded in $|z| \geqslant a_1$, $\alpha_1 \leqslant \text{ph}\, z \leqslant \beta_1$, that is, for each integer m, there exists a constant A_m such that

$$|\phi_m(z)| < A_m$$

there.

Since $f(z)$ is regular in R, it is, by definition, differentiable, and hence $\phi_m(z)$ is regular in R, and

$$f'(z) = -\sum_{n=1}^{m-1} \frac{na_n}{z^{n+1}} + \frac{\phi_m'(z)}{z^m} - \frac{m\phi_m(z)}{z^{m+1}}$$

$$= -\sum_{n=2}^{m-1} \frac{(n-1)\,a_{n-1}}{z^n} + \frac{\psi_m(z)}{z^m},$$

where $$\psi_m(z) = \phi_m'(z) - (m-1)\,a_{m-1} - \frac{m\phi_m(z)}{z}.$$

We have to show that $\psi_m(z)$ is bounded in any closed sector $\alpha_2 \leqslant \text{ph}\, z \leqslant \beta_2$, contained in R. Evidently it suffices to show that $\phi_m'(z)$ is bounded.

Given α_2, β_2, we choose α_1 and β_1 so that

$$\alpha < \alpha_1 < \alpha_2 < \beta_2 < \beta_1 < \beta.$$

Then $|\phi_m(z)| < A_m$ in $\alpha_1 \leqslant \text{ph}\, z \leqslant \beta_1$. We can choose a positive number δ so that, if ζ lies in $\alpha_2 \leqslant \text{ph}\, \zeta \leqslant \beta_2$, the circle c whose equation is $|z - \zeta| = \delta|\zeta|$ lies in $\alpha_1 \leqslant \text{ph}\, z \leqslant \beta_1$. Hence

$$\phi_m'(\zeta) = \frac{1}{2\pi i}\int_c \frac{\phi_m(z)}{(z-\zeta)^2}\,dz = \frac{1}{2\pi}\int_0^{2\pi} \frac{\phi_m(\zeta + \delta\zeta\, e^{\theta i})}{\delta\zeta\, e^{\theta i}}\,d\theta,$$

and so $$|\phi_m'(\zeta)| \leqslant \frac{A_m}{\delta|\zeta|} \leqslant \frac{A_m}{\delta a}.$$

This proves the result.

An asymptotic power series expansion of an analytic function usually holds in a sectorial region. Such a function may possess different asymptotic expansions in different sectors, an effect known as the Stokes Phenomenon.

(viii) *If $f(z)$ is a one-valued function regular in $|z| \geqslant a$ and if*

$$f(z) \sim \sum_0^\infty \frac{a_n}{z^n}$$

as $z \to \infty$ for all values of ph z, *the asymptotic power series is convergent with sum $f(z)$ for all sufficiently large values of $|z|$.*

Let R_1 be any number greater than a. Then $f(z)$ has a Laurent expansion

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n$$

convergent in $|z| \geqslant R_1$, where

$$c_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z^{n+1}}\,dz,$$

Γ being any circle $|z| = R$ where $R > R_1$. Since $f(z)$ tends to a_0 as $z \to \infty$, it is bounded; thus there exists a constant M such that $|f(z)| \leqslant M$ when $|z| \geqslant a$. When $n > 0$,

$$|c_n| \leqslant \frac{M}{R^n}.$$

But R can be as large as we please. Hence $c_n = 0$ when $n > 0$, and therefore

$$f(z) = \sum_0^\infty c_{-n} z^{-n},$$

the series being convergent when $|z| \geqslant R_1$. But

$$f(z) \sim \sum_0^\infty a_n z^{-n}$$

as $z \to \infty$, and so

$$a_0 = \lim_{z\to\infty} f(z) = c_0,$$

$$a_1 = \lim_{z\to\infty} \{f(z) - a_0\}\Big/\frac{1}{z} = c_{-1},$$

$$a_2 = \lim_{z\to\infty} \left\{f(z) - a_0 - \frac{a_1}{z}\right\}\Big/\frac{1}{z^2} = c_{-2};$$

in general $a_n = c_{-n}$. Hence the asymptotic power series expansion of $f(z)$ is convergent.

CHAPTER 3

INTEGRATION BY PARTS

5. The Incomplete Gamma Function

One of the simplest ways of finding the asymptotic expansion of a function defined by a definite integral is the method of integration by parts. The successive terms of the asymptotic series are produced by repeated integration by parts, and the asymptotic character of the series is then proved by examining the remainder, which is in the form of a definite integral. The field of application of the method is rather restricted, and it is difficult to formulate precise theorems of any degree of generality. Instead of attempting this, we try to make the idea clear by discussing particular examples.

As a first example, we take the Incomplete Gamma Function defined by

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} \, dt,$$

where x and a are positive. A series suitable for calculation when x is small can be deduced at once by expanding the exponential function and integrating term by term. This series

$$\gamma(a, x) = \sum_0^\infty \frac{(-1)^n}{a+n} \frac{x^{n+a}}{n!}$$

converges for all positive values of x, but is of little use for numerical work. For example, if $x = 10$, $a = \frac{1}{2}$, the largest term, corresponding to $n = 8$, is about 923, yet $\gamma(\frac{1}{2}, 10)$ is equal to $\sqrt{\pi}$ with an error of the order of 10^{-5}.

When x is large and positive, it is better to consider the function

$$\Gamma(a, x) = \Gamma(a) - \gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} \, dt,$$

where the integral is convergent for all values of the parameter a. If we integrate by parts once, we get

$$\Gamma(a, x) = e^{-x} x^{a-1} + (a-1) \, \Gamma(a-1, x).$$