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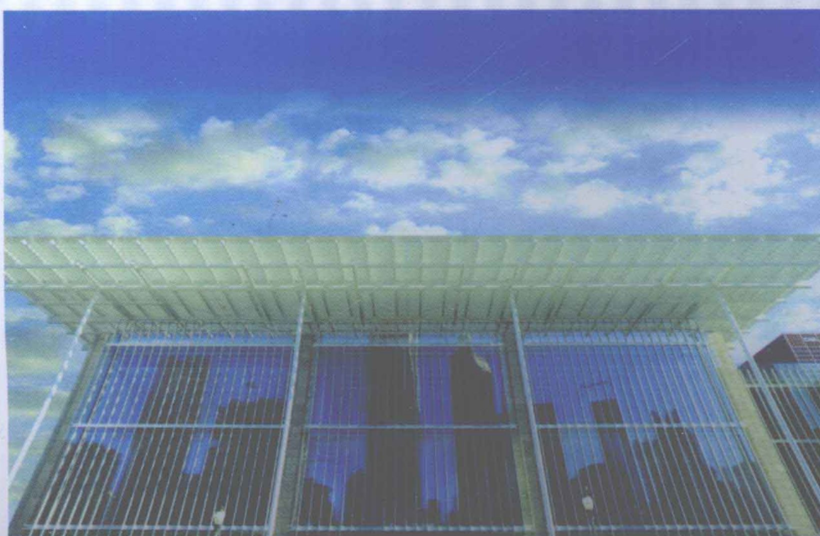
线性代数

（英文版·第8版）



LINEAR ALGEBRA

WITH APPLICATIONS

EIGHTH EDITION

STEVEN J. LEON

（美）Steven J. Leon 著
马萨诸塞大学达特茅斯分校

机械工业出版社
China Machine Press

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Preface

I am pleased to see the text reach its eighth edition. The continued support and enthusiasm of the many users has been most gratifying. Linear algebra is more exciting now than at almost any time in the past. Its applications continue to spread to more and more fields. Largely due to the computer revolution of the last half century, linear algebra has risen to a role of prominence in the mathematical curriculum rivaling that of calculus. Modern software has also made it possible to dramatically improve the way the course is taught. I teach linear algebra every semester and continue to seek new ways to optimize student understanding. For this edition, every chapter has been carefully scrutinized and enhanced. Additionally, many of the revisions in this edition are due to the helpful suggestions received from users and reviewers. Consequently, this new edition, while retaining the essence of previous editions, incorporates a variety of substantive improvements.

What's New in the Eighth Edition?

1. New Section on Matrix Arithmetic
   One of the longer sections in the previous edition was the section on matrix algebra in Chapter 1. The material in that section has been expanded further for the current edition. Rather than include an overly long revised section, we have divided the material into sections titled *Matrix Arithmetic* and *Matrix Algebra.*
2. New Exercises
   After seven editions it was quite a challenge to come up with additional original exercises. However, the eighth edition has more than 130 new exercises.
3. New Subsections and Applications
   A new subsection on cross products has been included in Section 3 of Chapter 2. A new application to Newtonian Mechanics has also been added to that section. In Section 4 of Chapter 6 (Hermitian Matrices), a new subsection on the *Real Schur Decomposition* has been added.
4. New and Improved Notation
   The standard notation for the jth column vector of a matrix A is $\mathbf{a}_j$, however, there seems to be no universally accepted notation for row vectors. In the MATLAB package, the ith row of A is denoted by $A(i, :)$. In previous editions of this book we used a similar notation $\mathbf{a}(i, :)$; however, this notation seems somewhat artificial. For this edition we use the same notation as a column vector except we put a horizontal arrow above the letter to indicate that the vector is a row vector (a horizontal array) rather than a column vector (a vertical array). Thus the ith row of A is now denoted by $\vec{\mathbf{a}}_i$.

   We have also introduced improved notation for the standard Euclidean vector spaces and their complex counterparts. We now use the symbols $\mathbb{R}^n$ and $\mathbb{C}^n$ in place of the R^n and C^n notation used in earlier editions.

5. Special Web Site and Supplemental Web Materials
Pearson has developed a special Web site to accompany the 8th edition:

www.pearsonhighered.com/leon

This site includes a host of materials for both students and instructors.

Overview of Text

This book is suitable for either a sophomore-level course or for a junior/senior-level course. The student should have some familiarity with the basics of differential and integral calculus. This prerequisite can be met by either one semester or two quarters of elementary calculus.

If the text is used for a sophomore-level course, the instructor should probably spend more time on the early chapters and omit many of the sections in the later chapters. For more advanced courses, a quick review of many of the topics in the first two chapters and then a more complete coverage of the later chapters would be appropriate. The explanations in the text are given in sufficient detail so that beginning students should have little trouble reading and understanding the material. To further aid the student, a large number of examples have been worked out completely. Additionally, computer exercises at the end of each chapter give students the opportunity to perform numerical experiments and try to generalize the results. Applications are presented throughout the book. These applications can be used to motivate new material or to illustrate the relevance of material that has already been covered.

The text contains all the topics recommended by the National Science Foundation (NSF) sponsored Linear Algebra Curriculum Study Group (LACSG) and much more. Although there is more material than can be covered in a one-quarter or one-semester course, I feel that it is easier for an instructor to leave out or skip material than it is to supplement a book with outside material. Even if many topics are omitted, the book should still provide students with a feeling for the overall scope of the subject matter. Furthermore, many students may use the book later as a reference and consequently may end up learning many of the omitted topics on their own.

In the next section of this preface a number of outlines are provided for one-semester courses at either the sophomore level or the junior/senior level and with either a matrix-oriented emphasis or a slightly more theoretical emphasis. To further aid the instructor in the choice of topics, three sections have been designated as optional and are marked with a dagger in the table of contents. These sections are not prerequisites for any of the following sections in the book. They may be skipped without any loss of continuity.

Ideally the entire book could be covered in a two-quarter or two-semester sequence. Although two semesters of linear algebra has been recommended by the LACSG, it is still not practical at many universities and colleges. At present there is no universal agreement on a core syllabus for a second course. Indeed, if all of the topics that instructors would like to see in a second course were included in a single volume, it would be a weighty book. An effort has been made in this text to cover all of the basic linear algebra topics that are necessary for modern applications. Furthermore, two additional chapters for a second course are available for downloading from the Internet. See the special Pearson Web page discussed earlier: *www.pearsonhighered.com/leon*.

Suggested Course Outlines

I. Two-Semester Sequence: In a two semester sequence it is possible to cover all 39 sections of the book. Additional flexibility is possible by omitting any of the three optional sections in Chapters 2, 5, and 6. One could also include an extra lecture demonstrating how to use the MATLAB software.

II. One-Semester Sophomore-Level Course

A. A Basic Sophomore-Level Course

Chapter 1	Sections 1–6	7 lectures
Chapter 2	Sections 1–2	2 lectures
Chapter 3	Sections 1–6	9 lectures
Chapter 4	Sections 1–3	4 lectures
Chapter 5	Sections 1–6	9 lectures
Chapter 6	Sections 1–3	4 lectures
	Total	35 lectures

B. The LACSG Matrix Oriented Course: The core course recommended by the Linear Algebra Curriculum Study Group involves only the Euclidean vector spaces. Consequently, for this course you should omit Section 1 of Chapter 3 (on general vector spaces) and all references and exercises involving function spaces in Chapters 3 to 6. All of the topics in the LACSG core syllabus are included in the text. It is not necessary to introduce any supplementary materials. The LACSG recommended 28 lectures to cover the core material. This is possible if the class is taught in lecture format with an additional recitation section meeting once a week. If the course is taught without recitations, it is my feeling that the following schedule of 35 lectures is perhaps more reasonable.

Chapter 1	Sections 1–6	7 lectures
Chapter 2	Sections 1–2	2 lectures
Chapter 3	Sections 2–6	7 lectures
Chapter 4	Section 1–3	2 lectures
Chapter 5	Sections 1–6	9 lectures
Chapter 6	Sections 1, 3–5	8 lectures
	Total	35 lectures

III. One-Semester Junior/Senior-Level Courses: The coverage in an upper division course is dependent on the background of the students. Below are two possible courses with 35 lectures each.

A. Course 1

Chapter 1	Sections 1–6	6 lectures
Chapter 2	Sections 1–2	2 lectures
Chapter 3	Sections 1–6	7 lectures
Chapter 5	Sections 1–6	9 lectures
Chapter 6	Sections 1–7 Section 8 if time allows	10 lectures
Chapter 7	Section 4	1 lecture
	Total	35 lectures

B. Course 2

Review of Topics in Chapters 1–3		5 lectures
Chapter 4	Sections 1–3	2 lectures
Chapter 5	Sections 1–6	10 lectures
Chapter 6	Sections 1–7 Section 8 if time allows	11 lectures
Chapter 7	Sections 4–7 If time allows, Sections 1–3	7 lectures
	Total	35 lectures

Computer Exercises

This edition contains a section of computing exercises at the end of each chapter. These exercises are based on the software package MATLAB. The MATLAB Appendix in the book explains the basics of using the software. MATLAB has the advantage that it is a powerful tool for matrix computations and yet it is easy to learn. After reading the Appendix, students should be able to do the computing exercises without having to refer to any other software books or manuals. To help students get started we recommend one 50 minute classroom demonstration of the software. The assignments can be done either as ordinary homework assignments or as part of a formally scheduled computer laboratory course.

Another source of MATLAB exercises for linear algebra is the ATLAST book which is available as a companion manual to supplement this book. (See the list of supplementary materials in the next section of this preface.)

While the course can be taught without any reference to the computer, we believe that computer exercises can greatly enhance student learning and provide a new dimension to linear algebra education. The Linear Algebra Curriculum Study Group has recommended that technology be used for a first course in linear algebra. and this view is generally accepted throughout the greater mathematics community.

Supplementary Materials

Additional Chapters

Two supplemental chapters for this book may be downloaded using links from my home page: *www.umassd.edu/cas/mathematics/people/leon* or from the Pearson Web site.

- Chapter 8. Iterative Methods
- Chapter 9. Canonical Forms

My home page also contains a link to the errata list for this textbook.

Web Supplements

The Pearson Web site for this book has an impressive collection of supplementary materials including links to the two supplementary chapters that were previously discussed. The URL for the Web site is:

www.pearsonhighered.com/leon

Companion Books

A *Student Study Guide* has been developed to accompany this textbook. A number of MATLAB and Maple computer manuals are also available as companion books. Instructors wishing to use one of the companion manuals along with the textbook can order both the book and the manual for their classes and have each pair bundled together in a shrink-wrapped package. These packages are available for classes at special rates that are comparable to the price of ordering the textbook alone. Thus, when students buy the textbook, they get the manual at little or no extra cost. To obtain information about the companion packages available, instructors should either consult their Pearson sales representative or search the instructor section of the Pearson higher education Web site (*www.pearsonhighered.com*). The following is a list of some of the companion books being offered as bundles with this textbook.

- *Student Guide to Linear Algebra with Applications*, ISBN 0-13-600930-1. The manual is available to students as a study tool to accompany this textbook. The manual summarizes important theorems, definitions, and concepts presented in the textbook. It provides solutions to some of the exercises and hints and suggestions on many other exercises.
- *ATLAST Computer Exercises for Linear Algebra, Second edition*, ISBN 0-13-101121-9.
  ATLAST (Augmenting the Teaching of Linear Algebra through the use of Software Tools) was an NSF sponsored project to encourage and facilitate the use of software in the teaching of linear algebra. During a five year period, 1992–1997, the ATLAST Project conducted 18 faculty workshops using the MATLAB software package. Participants in those workshops designed computer exercises, projects, and lesson plans for software-based teaching of linear algebra. A selection of these materials was first published as a manual in 1997. That manual was greatly expanded for the second edition published in 2003. Each of the eight chapters in the second edition contains a section of short exercises and a section of longer projects.
  The collection of software tools (M-files) developed to accompany the ATLAST book may be downloaded from the ATLAST Web site:

  www.umassd.edu/specialprograms/atlast

  Additionally, Mathematica users can download the collection of *ATLAST Mathematica Notebooks* that has been developed by Richard Neidinger.
- *Linear Algebra Labs with MATLAB: 3rd ed.* by David Hill and David Zitarelli
- *Visualizing Linear Algebra using Maple*, by Sandra Keith
- *Maple Supplement for Linear Algebra*, by John Maloney
- *Understanding Linear Algebra Using MATLAB*, by Irwin and Margaret Kleinfeld

Acknowledgments

I would like to express my gratitude to the long list of reviewers that have contributed so much to all eight editions of this book. Thanks also to the many users who have sent in comments and suggestions.

A special thanks is due to the reviewers of the eighth edition:

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Thomas Riedel, University of Louisville
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Many of the revisions and new exercises in this latest edition are a direct result of their comments and suggestions.

Special thanks to Mathematics Editor Caroline Celano for her help in planning the new edition and to Project Manager Robert Merenoff for his work on coordinating production of the book. Thanks to accuracy checker Tom Wegleitner and to the entire editorial, production, and sales staff at Pearson for all their efforts.

I would like to acknowledge the contributions of Gene Golub and Jim Wilkinson. Most of the first edition of the book was written in 1977–1978 while I was a Visiting Scholar at Stanford University. During that period I attended courses and lectures on numerical linear algebra given by Gene Golub and J. H. Wilkinson. Those lectures have greatly influenced this book. Finally, I would like to express my gratitude to Germund Dahlquist for his helpful suggestions on earlier editions of this book. Although Gene Golub, Jim Wilkinson, and Germund Dahlquist are no longer with us, they continue to live on in the memories of their friends.

Steven J. Leon
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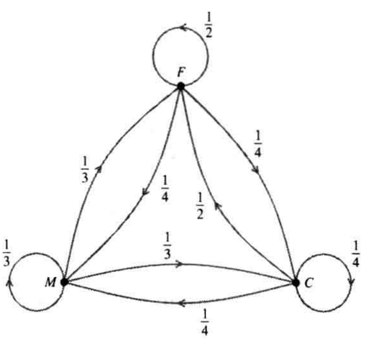
Contents

† Optional sections. These sections are not prerequisites for any other sections of the book.

* Web: The supplemental Chapters 8 and 9 can be downloaded from the internet. See the section of the Preface on supplemental Web materials.

CHAPTER

1



Matrices and Systems of Equations

Probably the most important problem in mathematics is that of solving a system of linear equations. Well over 75 percent of all mathematical problems encountered in scientific or industrial applications involve solving a linear system at some stage. By using the methods of modern mathematics, it is often possible to take a sophisticated problem and reduce it to a single system of linear equations. Linear systems arise in applications to such areas as business, economics, sociology, ecology, demography, genetics, electronics, engineering, and physics. Therefore, it seems appropriate to begin this book with a section on linear systems.

1.1 Systems of Linear Equations

A *linear equation in n unknowns* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers and $x_1, x_2, \ldots, x_n$ are variables. A *linear system* of m equations in n unknowns is then a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

where the a_{ij}'s and the b_i's are all real numbers. We will refer to systems of the form (1) as $m \times n$ linear systems. The following are examples of linear systems:

(a) $\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned}$ **(b)** $\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$ **(c)** $\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 \quad &= 4 \end{aligned}$

System **(a)** is a 2×2 system, **(b)** is a 2×3 system, and **(c)** is a 3×2 system.

By a solution of an $m \times n$ system, we mean an ordered n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ that satisfies all the equations of the system. For example, the ordered pair $(1, 2)$ is a solution of system **(a)**, since

$$\begin{aligned} 1 \cdot (1) + 2 \cdot (2) &= 5 \\ 2 \cdot (1) + 3 \cdot (2) &= 8 \end{aligned}$$

The ordered triple $(2, 0, 0)$ is a solution of system **(b)**, since

$$\begin{aligned} 1 \cdot (2) - 1 \cdot (0) + 1 \cdot (0) &= 2 \\ 2 \cdot (2) + 1 \cdot (0) - 1 \cdot (0) &= 4 \end{aligned}$$

Actually, system **(b)** has many solutions. If α is any real number, it is easily seen that the ordered triple $(2, \alpha, \alpha)$ is a solution. However, system **(c)** has no solution. It follows from the third equation that the first coordinate of any solution would have to be 4. Using $x_1 = 4$ in the first two equations, we see that the second coordinate must satisfy

$$\begin{aligned} 4 + x_2 &= 2 \\ 4 - x_2 &= 1 \end{aligned}$$

Since there is no real number that satisfies both of these equations, the system has no solution. If a linear system has no solution, we say that the system is *inconsistent*. If the system has at least one solution, we say that it is *consistent*. Thus, system **(c)** is inconsistent, while systems **(a)** and **(b)** are both consistent.

The set of all solutions of a linear system is called the *solution set* of the system. If a system is inconsistent, its solution set is empty. A consistent system will have a nonempty solution set. To solve a consistent system, we must find its solution set.

2×2 Systems

Let us examine geometrically a system of the form

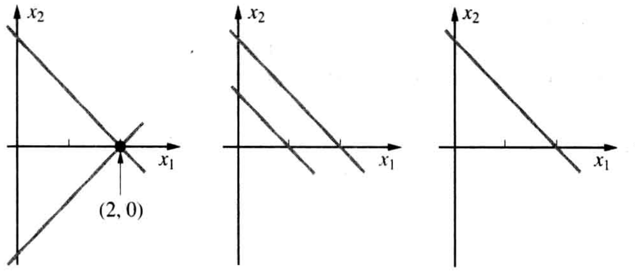
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

Each equation can be represented graphically as a line in the plane. The ordered pair (x_1, x_2) will be a solution of the system if and only if it lies on both lines. For example, consider the three systems

$$\textbf{(i)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \end{aligned} \qquad \textbf{(ii)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned} \qquad \textbf{(iii)}\ \begin{aligned} x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}$$

The two lines in system (i) intersect at the point $(2, 0)$. Thus, $\{(2, 0)\}$ is the solution set of (i). In system (ii) the two lines are parallel. Therefore, system (ii) is inconsistent and hence its solution set is empty. The two equations in system (iii) both represent the same line. Any point on this line will be a solution of the system (see Figure 1.1.1).

In general, there are three possibilities: the lines intersect at a point, they are parallel, or both equations represent the same line. The solution set then contains either one, zero, or infinitely many points.



(i) (ii) (iii)

Figure 1.1.1.

The situation is the same for $m \times n$ systems. An $m \times n$ system may or may not be consistent. If it is consistent, it must have either exactly one solution or infinitely many solutions. These are the only possibilities. We will see why this is so in Section 2 when we study the row echelon form. Of more immediate concern is the problem of finding all solutions of a given system. To tackle this problem, we introduce the notion of *equivalent systems.*

Equivalent Systems

Consider the two systems

$$\textbf{(a)}\quad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 &= 3 \\ 2x_3 &= 4 \end{aligned} \qquad\qquad \textbf{(b)}\quad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

System **(a)** is easy to solve because it is clear from the last two equations that $x_2 = 3$ and $x_3 = 2$. Using these values in the first equation, we get

$$\begin{aligned} 3x_1 + 2 \cdot 3 - \ 2 &= -2 \\ x_1 &= -2 \end{aligned}$$

Thus, the solution of the system is $(-2, 3, 2)$. System **(b)** seems to be more difficult to solve. Actually, system **(b)** has the same solution as system **(a)**. To see this, add the first two equations of the system:

$$\begin{array}{r} 3x_1 + 2x_2 - x_3 = -2 \\ -3x_1 - x_2 + x_3 = 5 \\ \hline x_2 = 3 \end{array}$$

If (x_1, x_2, x_3) is any solution of **(b)**, it must satisfy all the equations of the system. Thus, it must satisfy any new equation formed by adding two of its equations. Therefore, x_2 must equal 3. Similarly, (x_1, x_2, x_3) must satisfy the new equation formed by subtracting the first equation from the third:

$$\begin{array}{r} 3x_1 + 2x_2 + \ x_3 = 2 \\ 3x_1 + 2x_2 - \ x_3 = -2 \\ \hline 2x_3 = 4 \end{array}$$

Therefore, any solution of system **(b)** must also be a solution of system **(a)**. By a similar argument, it can be shown that any solution of **(a)** is also a solution of **(b)**. This can be done by subtracting the first equation from the second:

$$\begin{array}{rcl} x_2 \qquad\quad &=& 3 \\ 3x_1 + 2x_2 - x_3 &=& -2 \\ \hline -3x_1 - x_2 + x_3 &=& 5 \end{array}$$

Then add the first and third equations:

$$\begin{array}{rcl} 3x_1 + 2x_2 - x_3 &=& -2 \\ 2x_3 &=& 4 \\ \hline 3x_1 + 2x_2 + x_3 &=& 2 \end{array}$$

Thus, (x_1, x_2, x_3) is a solution of system **(b)** if and only if it is a solution of system **(a)**. Therefore, both systems have the same solution set, $\{(-2, 3, 2)\}$.

Definition

> Two systems of equations involving the same variables are said to be **equivalent** if they have the same solution set.

Clearly, if we interchange the order in which two equations of a system are written, this will have no effect on the solution set. The reordered system will be equivalent to the original system. For example, the systems

$$\begin{array}{rcl} x_1 + 2x_2 &=& 4 \\ 3x_1 - x_2 &=& 2 \\ 4x_1 + x_2 &=& 6 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} 4x_1 + x_2 &=& 6 \\ 3x_1 - x_2 &=& 2 \\ x_1 + 2x_2 &=& 4 \end{array}$$

both involve the same three equations and, consequently, they must have the same solution set.

If one equation of a system is multiplied through by a nonzero real number, this will have no effect on the solution set, and the new system will be equivalent to the original system. For example, the systems

$$\begin{array}{rcl} x_1 + x_2 + x_3 &=& 3 \\ -2x_1 - x_2 + 4x_3 &=& 1 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} 2x_1 + 2x_2 + 2x_3 &=& 6 \\ -2x_1 - x_2 + 4x_3 &=& 1 \end{array}$$

are equivalent.

If a multiple of one equation is added to another equation, the new system will be equivalent to the original system. This follows since the n-tuple $(x_1, \ldots, x_n)$ will satisfy the two equations

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j \end{aligned}$$

if and only if it satisfies the equations

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ (a_{j1} + \alpha a_{i1})x_1 + \cdots + (a_{jn} + \alpha a_{in})x_n &= b_j + \alpha b_i \end{aligned}$$

To summarize, there are three operations that can be used on a system to obtain an equivalent system:

I. The order in which any two equations are written may be interchanged.

II. Both sides of an equation may be multiplied by the same nonzero real number.

III. A multiple of one equation may be added to (or subtracted from) another.

Given a system of equations, we may use these operations to obtain an equivalent system that is easier to solve.

$n \times n$ Systems

Let us restrict ourselves to $n \times n$ systems for the remainder of this section. We will show that if an $n \times n$ system has exactly one solution, then operations **I** and **III** can be used to obtain an equivalent "strictly triangular system."

Definition A system is said to be in **strict triangular form** if, in the kth equation, the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero $(k = 1, \ldots, n)$.

EXAMPLE 1 The system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

is in strict triangular form, since in the second equation the coefficients are 0, 1, -1, respectively, and in the third equation the coefficients are 0, 0, 2, respectively. Because of the strict triangular form, the system is easy to solve. It follows from the third equation that $x_3 = 2$. Using this value in the second equation, we obtain

$$x_2 - 2 = 2 \qquad \text{or} \qquad x_2 = 4$$

Using $x_2 = 4$, $x_3 = 2$ in the first equation, we end up with

$$\begin{aligned} 3x_1 + 2 \cdot 4 + 2 &= 1 \\ x_1 &= -3 \end{aligned}$$

Thus, the solution of the system is $(-3, 4, 2)$. ∎

Any $n \times n$ strictly triangular system can be solved in the same manner as the last example. First, the nth equation is solved for the value of x_n. This value is used in the $(n-1)$st equation to solve for x_{n-1}. The values x_n and x_{n-1} are used in the $(n-2)$nd equation to solve for x_{n-2}, and so on. We will refer to this method of solving a strictly triangular system as *back substitution*.