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h and *hp* Finite Element Discretizations

Vadim Glebovich Korneev

Ulrich Langer

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Vadim Glebovich Korneev

St. Petersburg State University, Russia

St. Petersburg State Polytechnical University, Russia

Ulrich Langer

Johannes Kepler University Linz, Austria

Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Linz, Austria

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Dedicated to our parents
Ekaterina and Gleb Korneev,
Gertraude and Gottfried Langer

Preface

Domain decomposition (DD) methods provide powerful tools for constructing parallel numerical solution algorithms for large scale systems of algebraic equations arising from the discretization of partial differential equations. The Alternating Schwarz Method, proposed by H. Schwarz (1869) in order to prove the existence of harmonic functions with prescribed Dirichlet data on the boundary of complicated domains, and substructuring techniques, developed by engineers in the 60s of the preceding century in order to create faster procedures for the analysis of complex structures, are commonly accepted as the origins of modern DD methods. Thus, DD methods have been developed for a long time, but most extensively since the first international DD conference that was held at Paris in 1987. This concerns both the theory and the practical use of DD techniques for creating efficient application software for massive parallel computers. The advances in DD are well documented in the proceedings of the international DD conferences[1] since 1987 and numerous papers. The first textbook on DD methods was published by B.F. Smith, P.E. Bjørstad, and W. Gropp in 1996. In 1999, A. Quarteroni and A. Vali published a monograph on "Domain Decomposition Methods for Partial Differential Equations" that provides a deep analysis of various domain decomposition methods. The authors contributed chapter 22 on "Domain Decomposition Methods and Preconditioning" to the Encyclopedia of Computational Mechanics (2004) in which they summarize the state of the art until 2003. Since then many papers on DD techniques and their use in Scientific Computing have been published. In particular, the textbook by A. Toselli and O. Widlund (2005) is now the standard reference. Another textbook, covering many interesting applications of DD methods, was published by T.P.A. Mathew in 2008. C.

[1] `http://www.ddm.org/conferences.html`

Pechstein has recently published a monograph on Finite Element Tearing and Interconnecting (FETI) and Boundary Element Tearing and Interconnecting (BETI) methods for a special class of multiscale problems. Our book is different from the textbooks mentioned above and very different from Pechstein's monograph. We mainly discuss one special class of primal substructuring methods also called Dirichlet–Dirichlet DD methods, and we emphasize the peculiarities of their application to *hp* finite element equations. In particular, we discuss and analyze the inexact versions of these DD methods which lead to optimal or, at least, almost optimal complexity and high efficiency in practical applications. The optimization of several important components of DD algorithms for *hp* discretization was achieved only quite recently. These topics are not discussed or at least not sufficiently discussed in the books mentioned above. The contributions of the authors to this field mainly appeared in journal publications including joint papers. The reader will become familiar with inexact Dirichlet–Dirichlet DD methods enjoying optimal complexity and the techniques for the numerical analysis of these methods. Thus, we hope that readers interested in both practice and theory can benefit from our book. In particular, the book will open the possibility for the reader to use such DD methods in new fields of applications like multiphysics applications in Computational Sciences.

In this book, the reader can find a brief historical overview, the basic results of the general theory of domain and space decompositions as well as the description and analysis of practical DD algorithms. Elliptic problems with strongly jumping coefficients are daily met in the engineering practice. Numerical techniques should not deteriorate in efficiency when solving such problems. Partly for this reason, we concentrate on h and *hp* finite element discretizations of elliptic boundary value problems and their solution by Dirichlet–Dirichlet-type DD methods. Considerable attention is paid to DD methods or preconditioners for *hp* discretizations, significant advances in the development of which having been made in the last decades. The book will add new features to the understanding of the beauty and powerfulness of DD methods, and supply the reader with a variety of modern DD algorithms. More precisely, this book contains 9 chapters where we discuss the following topics.

In the introductory **Chapter 1**, we present a brief historical retrospective of the development of modern DD methods, primarily of the Dirichlet–Dirichlet types, and discuss its two origins, which are not simply artifacts, but retained their significance in our time. One of these origins is the iterative procedure introduced by H. Schwarz in 1869. It is called now Alternating Schwarz Method, and has led to various modern additive and multiplicative overlapping domain decomposition methods. We trace advances to two other classes of additive domain decomposition methods and methods with nonoverlapping subdomains of decomposition and, eventually, to the period of fast development of variety of DD techniques and their theory, started in the middle of the 80s. In the 60s, engineers developed Gaussian elimination algorithms, called substructuring algorithms, which is commonly considered as another origin of DD approaches. The "block scheme" of these algorithms is closely reflected, and received much more powerful filling in the subclass of Dirichlet–Dirichlet DD methods with nonoverlapping subdomains.

Chapter 2 is devoted to the *Fundamentals of Schwarz' methods.* The look at the DD method as a method based on the problem space decomposition played indispensable role in the creation of its theoretical foundations. Another feature of the modern view of DD method is that it amounts to the use of special DD preconditioners in frames of known iterative methods like the Preconditioned Conjugate Gradient (PCG) method. Apart from good relative condition numbers, such preconditioners should suggest cheap solvers for the systems of algebraic equations with these preconditioner as system matrices. For this reason, they are often termed as preconditioners-solvers. These basic facts and some of their consequences are illuminated with the use of two model problems, from which one is the scalar elliptic equation with the diagonal matrix of coefficients and the other is the system of linear elasticity equations. Convergence estimates for basic versions of DD method are formulated in terms of conditions imposed on the energy space decomposition and subspace preconditioning bilinear forms. In the modern literature, these conditions imposed on the decomposition are called stability conditions.

Chapter 3 deals with *Overlapping Domain Decomposition Methods.* A good illustration of usefulness of the FE space decomposition technique for the analysis of efficiency of DD preconditioners are overlapping DD methods. The sizes of overlap have two opposite effects: growth of the overlap improves the relative condition number of the DD preconditioner, but increases the computational overhead. We present estimates, sharp in some aspects, for the relative spectrum bounds explicitly depending on the size of overlap. They show that, for sufficiently regular domain decompositions, the relative condition number is bounded by a generic constant provided that the overlap is of the order of the diameters of the subdomains.

Chapter 4 considers *Nonoverlapping DD Methods for h FE Discretizations in 2d.* The structure of nonoverlapping DD methods of the Dirichlet–Dirichlet type, especially in 3d case, is more complex in comparison with the overlapping DD methods. In this chapter, this is illustrated for h discretizations of 2d elliptic problems and includes the discussion of the three main components composing such DD preconditioner. They are the local preconditioner-solvers for Dirichlet problems on subdomains of decomposition, preconditioner-solver for the interface subproblem, governed by the interface Schur complement matrix, and prolongation operators from inter-subdomain boundaries inside the subdomains. At present, all these components are rather well developed, and there is a number of good options for each. A few of them, which yield DD solvers of linear or almost linear computational complexity are presented in this chapter.

Chapter 5 is devoted to *BPS-type DD Preconditioners for 3d Elliptic Problems.* In fact, we consider the DD preconditioner of Bramble, Pasciak and Schatz, which is now usually referred to as BPS preconditioner in the literature. In this chapter, it is investigated under more general conditions, imposed on the domain decompositions, their subdomains and their FE discretizations, than in the renowned series of papers by J.H. Bramble, J.E. Pasciak and A.H. Schatz. In the analysis of convergence, we tried to follow these papers, where it was possible, although in several instances, it was necessary to change their proofs, in order to adapt them to a more general situation. The use of BPS-type preconditioner in PCG iterative procedure results in very efficient solvers, convenient for parallelization and incorporating fast solvers of other types in its main components. For sufficiently regular problems and triangulations the losses in the relative condition number are not significant, and there is a number of such choices for component solvers that in a whole DD solver will possess almost linear numerical complexity.

In **Chapter 6**, we consider *DD Algorithms for Discretizations with Chaotically Piecewise Variable Orthotropism.* One class of problems among the mentioned above, which is studied in this chapter, often met in practice and cause significant difficulties at numerical solution. A representative 2d model problem is described by the equation $-\nabla \cdot \boldsymbol{\rho} \nabla u = f$ with a positive diagonal coefficient matrix $\boldsymbol{\rho}$. Computational difficulties of implementation of DD method are caused by three factors. The coefficients on the diagonal of the matrix $\boldsymbol{\rho}$ are arbitrary positive numbers, different for each subdomain. The decomposition mesh is a nonuniform rectangular mesh with changing arbitrarily mesh sizes. The finite element mesh is a nonuniform rectangular mesh satisfying only one condition that it is uniform on each subdomain. For a rectangular domain, we present a DD solver, which under the assumption of a fixed number of subdomains is optimal with respect to the arithmetical work. More precisely, it requires $\mathcal{O}(N)$ arithmetic operations, where N denotes the number of unknowns. The solver remains optimal, if the number of subdomains grows with N, but not too fast. The key problem in constructing efficient DD algorithm for the described discretization appears in preconditioning of the Schur complement, arising after elimination of the internal degrees of freedom for the subdomains of the decomposition. It interferes with the problem of deriving boundary norms for harmonic functions on slim domains and related problem of boundary norms for discrete harmonic functions corresponding to orthotropic differential operators on slim domains, discretized by means of orthotropic meshes. These problems are studied for one subdomain in Section 7.1. It is worth noting, that there are many suggestions in the literature on the efficient preconditioning of the boundary Schur complement for the tensor product orthotropic discretizations on slim rectangles. However, their use for assembling the inter-subdomain boundary Schur complement preconditioner is problematic for the reason of their incompatibility of definite sort for the described domain decompositions. Therefore, special way of constructing such a Schur complement preconditioner must be implemented, which is presented in Section 6.4.

Chapter 7 is devoted to *Nonoverlapping DD Methods for hp Discretizations of 2d Elliptic Equations.* Two-dimensional elliptic problems are quite good examples for the introduction of basic techniques employed at the derivation of nonoverlapping DD methods for more general *hp* discretization. Two typical structures of DD preconditioner-solvers, resembling structures of the DD preconditioner-solvers for *h* discretizations in Chapter 4 are discussed. In one of them, the vertex unknowns are split from the others in the DD preconditioner. In the other one, which can be more efficient, an inexact iterative solver is used for approximation of the Schur complement related to the whole (global) interface problem. We derive the bounds for the relative condition numbers and discuss efficient components for the both versions.

Chapter 8 deals with *Fast Dirichlet Solvers for 2d Reference Elements.* Fast solvers for *p* reference elements stiffness and mass matrices are of crucial importance for the performance of DD solvers for *hp* finite element discretizations. To the best of the authors' knowledge, all such solvers were obtained on the basis of finite-difference preconditioners (equivalently, finite element preconditioners induced by first order finite elements) for the stiffness and mass matrices. For both the hierarchical-type and the spectral-type square reference elements, known solvers of linear and almost linear complexity can be divided into three groups: DD type methods, algebraic multigrid methods, and multilevel wavelet methods, *i.e.*, based on multilevel wavelet decompositions of the first order FE space. We consider representatives of all of them. Note that the operator extrapolation technique allows us to obtain fast multilevel wavelet solvers for local problems on faces, arising, *e.g.*, in Dirichlet–Dirichlet DD methods for 3d elliptic equations.

Chapter 9 is devoted to *Nonoverlapping Dirichlet–Dirichlet DD Methods for hp Discretizations of 3d Elliptic Equations.* Nonoverlapping Dirichlet–Dirichlet methods for *hp* discretizations of 3d elliptic problems are much more complex than their 2d counterparts. The chapter starts from the discussion of the general structure of DD solver, common features of main components and their interplay. Results on the relative condition number of DD preconditioner and its numerical complexity are formulated first in terms of general properties of precondioners-solvers and prolongations incorporated in the components. After that, a few specific solvers and prolongations for each component are formulated and studied. We also touch on some special cases, for instance, when discretizations possess features that are typical for the *hp* adaptivity and when incomplete finite elements are used. The presented solvers and prolongations lead to Dirichlet–Dirichlet DD preconditioner-solvers of almost linear complexity of the preconditioning operations for general second order elliptic equations in arbitrary sufficiently smooth domains with subdomainwise smooth coefficients. It is necessary to stress that the numerical complexity of preconditioning means the arithmetical cost of the PCG with this preconditioner, except for the cost required for the matrix-vector multiplications by the matrix of the finite element system of equations to be solved. The cost of such matrix-vector multiplications as well as the cost of computation of the finite element system stiffness matrix for the *hp*-version can be considerable and deserve a special study not attended in this book.

The authors would like to thank many friends and colleagues, whose interest, support and encouragement made this book possible. Special thanks go to M.M. Karchevskii for reading the manuscript at an early stage and for numerous useful suggestions for improving the text. Almost all numerical experiments, the results of which are included in the book, were performed by I.E. Anufriev whom we owe our sincere gratitude. We had many fruitful discussions with S. Beuchler, whose personal contributions to fast solvers for *hp*-version found significant reflection in this book. The idea to write this book was born during our first Research-in-Pairs (RiP) stay at Mathematical Research Institute of Oberwolfach (MFO) in 2002 when we were working on our contribution *"Domain Decomposition Methods and Preconditioning"* to the Encyclopedia of Computational Mechanics that was published by John Wiley & Sons in 2004. Our second RiP stay in August 2013 enabled us to finish the work on this book. We are very grateful to the MFO for supporting us through the RiP programme and for providing us with a perfect research environment. We would also like to thank the Johann Radon Institute for Computational and Applied Mathematics (RICAM) of Austrian Academy of Sciences that supports the visits of the first author at Linz. A great part of the research contributed to the book by the first author was completed under support of the Russian Fund of Basic Research, grants 08-01-00676-a and 11-01-00667-a. The second author was partly supported by the National Research Network *"Geometry and Simulation"* of the Austrian Science Fund FWF under the grant NFN S11703.

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