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CAM 8

Some Topics in Industrial and Applied Mathematics

Rolf Jeltsch
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editors

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Preface

On the occasion that the Officers' Meeting and the Board Meeting of ICIAM (International Council for Industrial and Applied Mathematics) was held in Shanghai from May 26 to May 27, 2006, many famous industrial and applied mathematicians gathered in Shanghai from different countries. The Shanghai Forum on Industrial and Applied Mathematics was organized from May 25 to May 26, 2006 at Shanghai Science Hall for the purpose of inviting some of them to present their recent results and discuss recent trends in industrial and applied mathematics. Sixteen invited lectures have been given for this activity. This volume collects the material covered by most of these lectures. It will be very useful for graduate students and researchers in industrial and applied mathematics.

The editors would like take this opportunity to express their sincere thanks to all the authors in this volume for their kind contribution. We are very grateful to the Shanghai Association for Science and Technology (SAST), Fudan University, the National Natural Science Foundation of China (NSFC), The China Society for Industrial and Applied Mathematics (CSIAM), the Shanghai Society for Industrial and Applied Mathmatics (SSIAM), the Institut Sino-Français de Mathématiques Appliquées (ISFMA) and the International Council for Industrial and Applied Mathematics (ICIAM) for their help and support. Our special thanks are also due to Mrs. Zhou Chunlian for her efficient assistance in editing this book.

Rolf Jeltsch, Ta-Tsien Li, Ian H. Sloan
April 2007

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A Continuation Method for a Class of Periodic Evolution Variational Inequalities

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Abstract

In this paper, using the Brouwer topological degree, we prove an existence result for finite variational inequalities. This approach is also applied to obtain the existence of periodic solutions for a class of evolution variational inequalities.

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1 Introduction

It has been well recognised that variational inequalities offer the right framework to consider numerous applied problems in various areas such as economics and engineering. Throughout the paper we consider $\mathbb{R}^n$ equipped with the usual Euclidean scalar product $\langle \cdot, \cdot \rangle$. We start by considering a variational inequality $\mathrm{VI}(\Lambda, \varphi)$ that is the problem of finding $\overline{x} \in \mathbb{R}^n$ such that:

$$\langle \Lambda(\overline{x}), v - \overline{x} \rangle + \varphi(v) - \varphi(\overline{x}) \geq 0, \quad \forall v \in \mathbb{R}^n.$$

In this formulation, $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a convex function. When the operator Λ under consideration is supposed to be coercive, existence results for the problem VI(Λ, φ) are well known in the setting of reflexive Banach spaces. This study was initiated by G. Stampacchia in the 60's and we refer to the contributions of J.L. Lions [15], Brézis [4] and Kinderlehrer & Stampacchia [12] for various results and references therein.

In the first part of this paper we develop an original approach essentially based on the use of the Brouwer topological degree to prove some results related to the existence of a solution to problem VI(Λ, φ).

Then, we study a first order evolution variational inequality, that is a differential inclusion of the form: find a T-periodic function $u \in C^0([0,T]; \mathbb{R}^n)$ such that:

$$\frac{du}{dt}(t) + F(u(t)) - f(t) \in -\partial\varphi(u(t)), \text{ a.e. } t \in [0,T],$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $f \in C^0([0,+\infty); \mathbb{R}^n)$ is such that: $\frac{df}{dt} \in L^1_{\text{loc}}(0,+\infty; \mathbb{R}^n)$, $T > 0$ is a prescribed period and $\partial\varphi$ is the convex subdifferential operator. This problem is studied by means of a continuation method. It is well known that the Brouwer topological degree plays a fundamental role in the theory of ordinary differential equations (ODE). M.A. Krasnosel'skii [13], [14] and H. Amann [3], developed a continuation method to compute this Brouwer topolgical degree associated to some gradient mapping (called the method of guiding function). This approach was useful for the study of the existence of periodic solutions for ODE's. Roughly speaking, if on some balls of $\mathbb{R}^n$ the Brouwer topological degree of the Poincaré translation operator (see e.g. [17]) associated to the ODE is different from zero, the problem has at least one periodic solution (for more details, references and possible extensions to the Leray-Schauder degree, we refer to the monograph of J. Mawhin [17]). With the emergence of many engineering disciplines and due to the lack of smoothness in many applications, it is not surprising that these classical mathematical tools require natural extension (for both analytical and numerical methods) to the class of unilateral dynamical systems. It is well known that the mathematical formulation of unilateral dynamical systems involves inequality constraints and hence contains natural nonsmoothness. In mechanical systems, this non-smoothness could have its origin in the environment of the system studied (e.g. case of contact) in the dry friction, or in the discontinuous control term. Recently, new analytical tools have been developed for the study of unilateral evolution problems (see e.g. [1], [2], [7], [8], [9] and references cited therein). The study of periodic solutions for evolution variational inequalities is also important. The Krasnol'skii's original approach for ODE, has known some extensions in order to obtain continuation methods for differential inclusions (see the article of L. Górniewicz [10] for more details and references). In the fourth section, we will be concerned with the existence of a T-periodic solution $u \in C^0([0,T];\mathbb{R}^n)$ such that:

$$\begin{gathered} \tfrac{du}{dt} \in L^\infty(0,T;\mathbb{R}^n); \\ u \text{ is right-differentiable on } [0,T); \\ u(0) = u(T); \\ \langle \tfrac{du}{dt}(t) + F(u(t)) - f(t), v - u(t)\rangle + \varphi(v) - \varphi(u(t)) \geq 0, \\ \forall v \in \mathbb{R}^n, \text{ a.e. } t \in [0,T]. \end{gathered} \tag{1}$$

Here $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $f \in C^0([0,+\infty[;\mathbb{R}^n)$ is such that: $\frac{df}{dt} \in L^1_{\text{loc}}(0,+\infty;\mathbb{R}^n)$ and $T > 0$ is a prescribed period.

We prove (Corollary 5.1) that if F and φ satisfy some growth condition (see (36)), then problem (1) has at least one periodic solution. This approach is also applied to obtain the existence of a T-periodic solution of a second order dynamical system of the form:

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) - F(t) \in -H_1\partial\Phi(H_1^T\dot{q}(t)), \tag{2}$$

where $q \in \mathbb{R}^m$ is the vector of generalized coordinate, $\Phi : \mathbb{R}^l \to \mathbb{R}$ is a convex function, $M \in \mathbb{R}^{m\times m}$ is a symmetric and positive definite matrix, $C \in \mathbb{R}^{m\times m}$ and $K \in \mathbb{R}^{m\times m}$ are given matrices and $H_1 \in \mathbb{R}^{m\times l}$ is a given matrix whose coefficients are determined by the directions of friction forces. The function $F \in C^0([0,+\infty);\mathbb{R}^m)$ is such that $\frac{dF}{dt} \in L^1_{\text{loc}}([0,+\infty);\mathbb{R}^m)$. The term $H_1\partial\Phi(H_1^T\cdot)$ is used to model the convex unilateral contact induced by friction forces. The paper is organized as follows: Section 2 contains some background materials on properties of the Brouwer topological degree and the concept of resolvent operator associated to a subdifferential operator. In Section 3, using an equivalent fixed point formulation as well as the Brouwer topological degree, we give some existence results for finite variational inequalities. In Section 4, we introduce the Poincaré operator associated to problem (1). Section 5 is devoted to the existence of a periodic solution of problem (1). In Section 6, we show that our approach could be applied to a special second order problem (2).

2 Brouwer topological degree and the resolvent operator $P_{\lambda,\varphi}$

It is well known that the degree theory is one of the most powerful tool in nonlinear analysis for the study of zeros of a continuous operator.

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset with boundary $\partial\Omega$ and $f \in C^1(\Omega;\mathbb{R}^n) \cap C^0(\overline{\Omega},\mathbb{R}^n)$. The Jacobian matrix of f at $x \in \Omega$ is defined by $f'(x) = (\partial_{x_i} f_j(x))_{1\leq i,j\leq n}$ and the Jacobian determinant of f at $x \in \Omega$ is defined by

$$J_f(x) = \det(f'(x)).$$

We set

$$A_f(\Omega) = \{x \in \Omega \ : \ J_f(x) = 0\}.$$

Observing that if $f^{-1}(0) \cap A_f(\Omega) = \emptyset$ and $0 \notin f(\partial\Omega)$, then the set $f^{-1}(0)$ is finite. The quantity $\sum_{x\in f^{-1}(0)} \mathrm{sign}\,(J_f(x))$ is therefore defined and is called the *Brouwer topological degree* of f with respect to Ω and 0 and is denoted by $\deg(f,\Omega,0)$. More generally, if $f \in C^0(\overline{\Omega};\mathbb{R}^n)$ and $0 \notin f(\partial\Omega)$, then the Brouwer topological degree of f with respect to Ω and 0, denoted by $\deg(f,\Omega,0)$, is well defined (see [16] for more details).

In the sequel, the scalar product on $\mathbb{R}^n$ is denoted as usual by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the associated norm. For $r > 0$, we set $\mathbb{B}_r := \{x \in \mathbb{R}^n : \|x\| < r\}$, and respectively $\bar{\mathbb{B}}_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$, for the open (respectively closed) unit ball with radius $r > 0$. As usual, we use the notation $\partial\mathbb{B}_r$ to denote the boundary $\bar{\mathbb{B}}_r \backslash \overset{o}{\mathbb{B}}_r$ of $\mathbb{B}_r$, that is $\{x \in \mathbb{R}^n : \|x\| = r\}$. If $f : \bar{\mathbb{B}}_r \to \mathbb{R}^n$ is continuous and $0 \notin f(\partial\mathbb{B}_r)$, then the Brouwer topological degree of f with respect to $\mathbb{B}_r$ and 0 is well-defined (see e.g. [16]) and denoted by $\deg(f,\mathbb{B}_r,0)$. Let us now recall some properties of the topological degree that we will use later.

P1. If $0 \notin f(\partial\mathbb{B}_r)$ and $\deg(f,\mathbb{B}_r,0) \neq 0$, then there exist $x \in \mathbb{B}_r$ such that $f(x) = 0$.

P2. Let $\varphi : [0,1] \times \bar{\mathbb{B}}_r \to \mathbb{R}^n; (\lambda,x) \to \varphi(\lambda,x)$, be continuous such that, for each $\lambda \in [0,1]$, one has $0 \notin \varphi(\lambda,\partial\mathbb{B}_r)$, then the map $\lambda \to \deg(\varphi(\lambda,.),\mathbb{B}_r,0)$ is constant on $[0,1]$.

P3. Let us denote by $id_{\mathbb{R}^n}$ the identity mapping on $\mathbb{R}^n$. We have

$$\deg(id_{\mathbb{R}^n},\mathbb{B}_r,0) = 1.$$

P4. If $0 \notin f(\partial\mathbb{B}_r)$ and $\alpha > 0$, then

$$\deg(\alpha f,\mathbb{B}_r,0) = \deg(f,\mathbb{B}_r,0)$$

and

$$\deg(-\alpha f,\mathbb{B}_r,0) = (-1)^n\deg(f,\mathbb{B}_r,0).$$

P5. If $0 \notin f(\partial\mathbb{B}_r)$ and f is odd on $\mathbb{B}_r$ (i.e., $f(-x) = -f(x), \quad \forall x \in \mathbb{B}_r$), then $\deg(f,\mathbb{B}_r,0)$ is odd.

P6. Let $f(x) = Ax - b$, with $A \in \mathbb{R}^{n\times n}$ a nonsingular matrix and $b \in \mathbb{R}^n$. Then $\deg(f, A^{-1}b + \mathbb{B}_r, 0) = \text{sign}(\det A) = \pm 1$.

Let $V \in C^1(\mathbb{R}^n;\mathbb{R})$ and suppose that there exists $r_0 > 0$ such that for every $r \geq r_0$, $0 \notin \nabla V(\partial\mathbb{B}_r)$. Then $\deg(\nabla V, \mathbb{B}_r, 0)$ is constant for $r \geq r_0$ and one defines the index of V at infinity "ind(V,∞)" by

$$\text{ind}(V,\infty) := \deg(\nabla V, \mathbb{B}_r, 0), \quad \forall r \geq r_0.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex mapping. It is well known that a) φ is continuous on $\mathbb{R}^n$; b) For all $x \in \mathbb{R}^n$, the convex subdifferential of φ at x is a nonempty compact and convex subset of $\mathbb{R}^n$ and defined by:

$$\partial\varphi(x) = \{w \in \mathbb{R}^n : \varphi(v) - \varphi(x) \geq \langle w, v-u\rangle, \quad \forall v \in \mathbb{R}^n\};$$

c) For all $x \in \mathbb{R}^n$, the directional derivative of φ at $x \in \mathbb{R}^n$ in the direction $\xi \in \mathbb{R}^n$, i.e.,

$$\varphi'(x;\xi) = \lim_{\alpha\downarrow 0} \frac{\varphi(x+\alpha\xi) - \varphi(x)}{\alpha}$$

exists (see e.g. [11] page 164). Since the subdifferential operator $\partial\varphi$ associated to φ, is maximal monotone (Brezis [5]), the operator $(I + \lambda\partial\varphi)^{-1}$ denoted by $P_{\lambda,\varphi}$ is a contraction eveywhere defined on $\mathbb{R}^n$, that is,

$$\|P_{\lambda,\varphi}(x) - P_{\lambda,\varphi}(y)\| \leq \|x-y\|, \quad \forall x,\ y \in \mathbb{R}^n.$$

This operator $P_{\lambda,\varphi}$ is called the resolvent of order $\lambda > 0$ associated to $\partial\varphi$ and for simplicity, we note P_φ instead of $P_{1,\varphi}$ when the parameter $\lambda = 1$. Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and consider the inequality problem: Find $\bar{x} \in \mathbb{R}^n$ such that

$$\langle \Lambda(\bar{x}), v - \bar{x}\rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in \mathbb{R}^n. \tag{3}$$

Clearly problem (3) is equivalent to the nonlinear equation: Find $\bar{x} \in \mathbb{R}^n$ such that

$$\bar{x} - P_\varphi(\bar{x} - \Lambda(\bar{x})) = 0. \tag{4}$$

In view of property P1 recalled earlier, it is important to compute the degree of the operator $id_{\mathbb{R}^n} - P_\varphi \circ (id_{\mathbb{R}^n} - \Lambda)$.

Remark 2.1. If $\bar{x}$ is a solution of problem (3), then

$$\langle \Lambda(\bar{x}), \xi\rangle + \varphi'(\bar{x};\xi) \geq 0, \quad \forall \xi \in \mathbb{R}^n.$$

Indeed, let $\bar{x}$ be a solution of (3). Let $\xi \in \mathbb{R}^n$ and $\alpha > 0$ be given. Setting $v = \bar{x} + \alpha\xi$ in (3), we get

$$\langle \Lambda(\bar{x}), \alpha\xi\rangle + \varphi(\bar{x} + \alpha\xi) - \varphi(\bar{x}) \geq 0.$$

Thus, for all $\alpha > 0$, we have

$$\langle \Lambda(\bar{x}), \xi \rangle + \frac{\varphi(\bar{x} + \alpha\xi) - \varphi(\bar{x})}{\alpha} \geq 0.$$

Taking the limit as $\alpha \downarrow 0$ we obtain

$$\langle \Lambda(\bar{x}), \xi \rangle + \varphi'(\bar{x}; \xi) \geq 0.$$

Example 2.1. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\varphi(x) = |x|, \quad \forall x \in \mathbb{R}.$$

We have

$$\partial\varphi(x) = \begin{cases} 1 & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and

$$P_\varphi(x) = (I + \partial\varphi)^{-1}(x) = \begin{cases} x - 1 & \text{if } x \geq 1 \\ 0 & \text{if } x \in [-1, 1] \\ x + 1 & \text{if } x \leq -1. \end{cases}$$

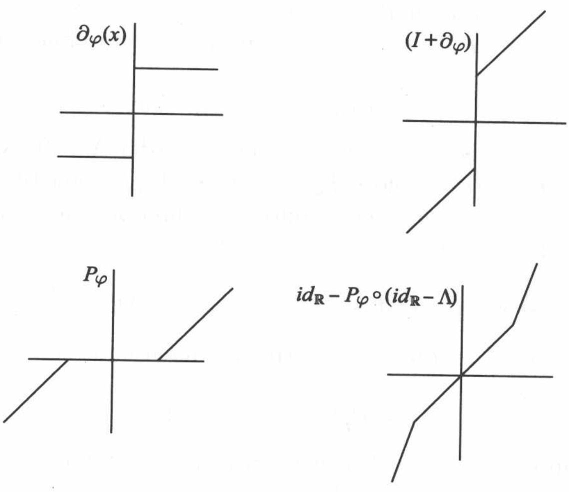

Figure 1 Example 2.1

Setting $\Lambda(x) = 2x$, we get

$$x - P_\varphi(x - \Lambda(x)) = \begin{cases} x & \text{if } |x| \leq 1 \\ 2x - 1 & \text{if } x \geq 1 \\ 2x + 1 & \text{if } x \leq -1. \end{cases}$$

We see that the operator $id_{\mathbb{R}} - P_\varphi \circ (id_{\mathbb{R}} - \Lambda)$ has a unique zero on $\mathbb{R}$.

Proposition 2.1. *Let $L > 0$ be given and assume that the mapping $G : [0, L] \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $(\lambda, y) \mapsto G(\lambda, y)$ is continuous on $[0, L] \times \mathbb{R}^n$. Then, the mapping*

$$(\lambda, y) \mapsto P_{\lambda,\varphi}(G(\lambda, y))$$

is continuous on $[0, L] \times \mathbb{R}^n$.

Proof: Let $\{y_n\} \subset \mathbb{R}^n$ and $\{\lambda_n\} \subset [0, L]$ be given sequences converging respectively to $y^* \in \mathbb{R}^n$ and $\lambda_n \to \lambda^* \in \mathbb{R}$ as $n \to +\infty$. We claim that the sequence $\{P_{\lambda_n,\varphi}(G(\lambda_n, y_n))\}$ tends to $P_{\lambda^*,\varphi}(G(\lambda^*, y^*))$ as $n \to +\infty$. Indeed, setting $x_n := P_{\lambda_n,\varphi}(G(\lambda_n, y_n))$ and $x^* := P_{\lambda^*,\varphi}(G(\lambda^*, y^*))$, we have

$$\langle x_n - G(\lambda_n, y_n), v - x_n\rangle + \lambda_n\varphi(v) - \lambda_n\varphi(x_n) \geq 0, \quad \forall v \in \mathbb{R}^n \tag{5}$$

and

$$\langle x^* - G(\lambda^*, y^*), v - x^*\rangle + \lambda^*\varphi(v) - \lambda^*\varphi(x^*) \geq 0, \quad \forall v \in \mathbb{R}^n. \tag{6}$$

Let us first check that the sequence $\{x_n\}$ is bounded. Indeed, suppose on the contrary that the sequence $\{\|x_n\|\}$ is unbounded. Setting $v := 0$ in (5), we obtain

$$-\langle x_n - G(\lambda_n, y_n), x_n\rangle + \lambda_n[\varphi(0) - \varphi(x_n)] \geq 0,$$

and thus

$$\|x_n\|^2 \leq \|G(\lambda_n, y_n)\|\|x_n\| + \lambda_n[\varphi(0) - \varphi(x_n)].$$

It results that for n large enough, $\|x_n\| \neq 0$ and

$$1 \leq \frac{\|G(\lambda_n, y_n)\|}{\|x_n\|} + \frac{\lambda_n}{\|x_n\|^2}[\varphi(0) - \varphi(x_n)]. \tag{7}$$

As for n large enough, $\frac{1}{\|x_n\|} \in (0, 1]$ we use the convexity of φ, to obtain

$$\varphi\Big(\frac{x_n}{\|x_n\|}\Big) \leq \frac{1}{\|x_n\|}\varphi(x_n) + \Big(1 - \frac{1}{\|x_n\|}\Big)\varphi(0).$$

Thus,

$$\frac{\varphi(0) - \varphi(x_n)}{\|x_n\|} \leq \varphi(0) - \varphi\Big(\frac{x_n}{\|x_n\|}\Big).$$

From (7), we get

$$1 \leq \frac{\|G(\lambda_n, y_n)\|}{\|x_n\|} + \lambda_n\left[\frac{\varphi(0) - \varphi(\frac{x_n}{\|x_n\|})}{\|x_n\|}\right]. \tag{8}$$

The sequence $\{\frac{x_n}{\|x_n\|}\}$ remains in the compact set $\partial\mathbb{B}_1$ and from the continuity of φ, we derive that the sequence $\{\varphi(\frac{x_n}{\|x_n\|})\}$ is bounded in $\mathbb{R}$. Hence,

$$\lim_{n\to+\infty} \frac{\varphi(\frac{x_n}{\|x_n\|})}{\|x_n\|} = 0.$$

Taking now the limit as $n \to +\infty$ in (8), we obtain the contradiction $1 \leq 0$. The sequence $\{x_n\}$ is thus bounded. Setting $v := x^*$ in (5) and $v := x_n$ in (6), we obtain the relations

$$\langle x_n - G(\lambda_n, y_n), x_n - x^*\rangle - \lambda_n\varphi(x^*) + \lambda_n\varphi(x_n) \leq 0 \tag{9}$$

and

$$-\langle x^* - G(\lambda^*, y^*), x_n - x^*\rangle - \lambda^*\varphi(x_n) + \lambda^*\varphi(x^*) \leq 0. \tag{10}$$

Thus

$$\begin{aligned}\|x_n - x^*\|^2 \leq \|G(\lambda_n, y_n) - G(\lambda^*, y^*)\|\|x_n - x^*\| \\ +(\lambda_n - \lambda^*)\varphi(x^*) + (\lambda^* - \lambda_n)\varphi(x_n).\end{aligned} \tag{11}$$

Using the continuity of φ and the boundeness of $\{x_n\}$, we get that the sequence $\{\varphi(x_n)\}_n$ is bounded in $\mathbb{R}$. Moreover $\|G(\lambda_n, y_n) - G(\lambda^*, y^*)\| \to 0$ and $(\lambda_n - \lambda^*) \to 0$ in $\mathbb{R}$ as $n \to +\infty$. Relation (11) yields that $x_n \to x^*$ in $\mathbb{R}^n$ as $n \to +\infty$. Hence the operator $(\lambda, y) \mapsto P_{\lambda,\varphi}(G(\lambda, y))$ is continuous, which completes the proof. □

Proposition 2.2. *Suppose that $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a convex function. If there exists a continuous mapping $H : \mathbb{R}^n \to \mathbb{R}^n$ and $r > 0$ such that*

$$\langle \Lambda(x), H(x)\rangle + \varphi'(x; H(x)) < 0, \quad \forall x \in \partial\mathbb{B}_r. \tag{12}$$

Then

$$\deg(H, \mathbb{B}_r, 0) = (-1)^n \deg(id_{\mathbb{R}^n} - P_\varphi(id_{\mathbb{R}^n} - \Lambda), \mathbb{B}_r, 0).$$

Proof: Let $h : [0,1] \times \bar{\mathbb{B}}_r \to \mathbb{R}^n; (\lambda, y) \mapsto h(\lambda, y) := y - P_{\lambda,\varphi}(y - \lambda\Lambda(y) + (1-\lambda)H(y))$. Proposition 2.1 ensures that h is continuous. Let us now check that $h(\lambda, x) \neq 0, \quad \forall x \in \partial\mathbb{B}_r$. Indeed, suppose on the contrary that there exists $x \in \partial\mathbb{B}_r$ and $\lambda \in [0,1]$ such that $h(\lambda, x) = 0$, that is

$$x = P_{\lambda,\varphi}(x - \lambda\Lambda(x) + (1-\lambda)H(x)).$$

We first remark that $\lambda \neq 0$. Indeed, if we suppose, on the contrary, that $\lambda = 0$, then $x = P_0(x + H(x)) = x + H(x)$. This yields $H(x) = 0$ which contradicts assumption (12). Thus $\lambda > 0$ and

$$\langle \lambda\Lambda(x) - (1-\lambda)H(x), v - x\rangle + \lambda\varphi(v) - \lambda\varphi(x) \geq 0, \quad \forall v \in \mathbb{R}^n.$$

It results that (see Remark 2.1):

$$\langle \lambda\Lambda(x) - (1-\lambda)H(x), \xi\rangle + \lambda\varphi'(x;\xi) \geq 0, \quad \forall \xi \in \mathbb{R}^n.$$

Setting $\xi := H(x)$, we obtain

$$\lambda[\langle \Lambda(x), H(x)\rangle + \varphi'(x;H(x))] \geq (1-\lambda)\|H(x)\|^2 \geq 0,$$

which contradicts assumption (12). Therefore,

$$\begin{aligned} \deg(id_{\mathbb{R}^n} - P_\varphi(id_{\mathbb{R}^n} - \Lambda), \mathbb{B}_r, 0) &= \deg(h(1,.), \mathbb{B}_r, 0) \\ &= \deg(h(0,.), \mathbb{B}_r, 0) \\ &= \deg(id_{\mathbb{R}^n} - P_0(id_{\mathbb{R}^n} + H), \mathbb{B}_r, 0) \\ &= \deg(-H, \mathbb{B}_r, 0) \\ &= (-1)^n \deg(H, \mathbb{B}_r, 0), \end{aligned}$$

which completes the proof. □

3 Some existence results for finite variational inequalities

As a direct consequence of Proposition 2.2, we have the following existence results for finite dimensional variational inequalities.

Theorem 3.1. *Suppose that* 1) $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous operator;* 2) $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is a convex function;* 3) *there exists* $r > 0$ *such that*

$$\langle \Lambda(x), x\rangle - \varphi'(x;-x) > 0, \quad \forall x \in \partial\mathbb{B}_r.$$

Then there exists $\bar{x} \in \mathbb{B}_r$ *such that*

$$\langle \Lambda(\bar{x}), v - \bar{x}\rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in \mathbb{R}^n.$$

Proof: Just apply Proposition 2.2 with $H := -id_{\mathbb{R}^n}$. Indeed, here we have

$$\langle \Lambda(x), H(x)\rangle + \varphi'(x;H(x)) = -\langle \Lambda(x), x\rangle + \varphi'(x;-x).$$

□

Theorem 3.2. *Suppose that* 1) $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous;* 2) $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is convex and Lipschitz continuous with Lipschitz constant* $K > 0$, i.e.,

$$| \varphi(x) - \varphi(y) | \leq K\|x - y\|, \quad \forall x, y \in \mathbb{R}^n;$$

3) *there exists* $r > 0$ *such that*

$$\|\Lambda(x)\| > K, \quad \forall x \in \partial\mathbb{B}_r,$$

and

$$\deg(\Lambda, \mathbb{B}_r, 0) \neq 0.$$

Then there exists $\bar{x} \in \mathbb{B}_r$ *such that*

$$\langle \Lambda(\bar{x}), v - \bar{x} \rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in \mathbb{R}^n.$$

Proof: Just apply Proposition 2.2 with $H := -\Lambda$. Indeed, we have

$$\begin{aligned} \langle \Lambda(x), H(x) \rangle + \varphi'(x; H(x)) &= -\|\Lambda(x)\|^2 + \varphi'(x; -\Lambda(x)) \\ &\leq -\|\Lambda(x)\|^2 + K\|\Lambda(x)\| \\ &= \|\Lambda(x)\|(K - \|\Lambda(x)\|). \end{aligned}$$

Therefore,

$$\langle \Lambda(x), H(x) \rangle + \varphi'(x; H(x)) < 0, \quad \forall x \in \partial\mathbb{B}_r.$$

Proposition 2.2 ensures that

$$\deg(id_{\mathbb{R}^n} - P_\varphi(id_{\mathbb{R}^n} - \Lambda), \mathbb{B}_r, 0) = (-1)^n \deg(H, \mathbb{B}_r, 0) = \deg(\Lambda, \mathbb{B}_r, 0) \neq 0.$$

Hence, there exists $\bar{x} \in \mathbb{B}_r$ such that $\bar{x} = P_\varphi(\bar{x} - \Lambda(\bar{x}))$. The conclusion follows. □

Theorem 3.3. *Suppose that*

1) $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and there exists* $r > 0$ *such that*

$$\langle \Lambda x, x \rangle > 0, \quad \forall x \in \partial\mathbb{B}_r \text{ and } \deg(\mathrm{id}_{\mathbb{R}^n} + \Lambda, \mathbb{B}_r, 0) \neq 0.$$

2) $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is a convex function satisfying,*

$$\varphi'(x; -x - \Lambda x) \leq 0, \quad \forall x \in \partial\mathbb{B}_r;$$

Then there exists $\bar{x} \in \mathbb{B}_r$ *such that*

$$\langle \Lambda(\bar{x}), v - \bar{x} \rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in \mathbb{R}^n.$$

Proof: Just take $H := -\mathrm{id}_{\mathbb{R}^n} - \Lambda$ in Proposition 2.2. Indeed, we have

$$\begin{aligned} &\langle \Lambda(x), H(x) \rangle + \varphi'(x; H(x)) \\ &= -\|\Lambda(x)\|^2 - \langle \Lambda(x), x \rangle + \varphi'(x; -x - \Lambda(x)) < 0, \quad \forall x \in \partial\mathbb{B}_r. \end{aligned}$$

According to Proposition 2.2, we have

$$\begin{aligned}\deg(id_{\mathbb{R}^n} - P_\varphi(id_{\mathbb{R}^n} - \Lambda), \mathbb{B}_r, 0) &= (-1)^n \deg(H, \mathbb{B}_r, 0)\\ &= \deg(id_{\mathbb{R}^n} + \Lambda, \mathbb{B}_r, 0) \neq 0.\end{aligned}$$

Hence, there exists $\bar{x} \in \mathbb{B}_r$ such that $\bar{x} = P_\varphi(\bar{x} - \Lambda(\bar{x}))$. The conclusion follows. □

Corollary 3.1. *Let $f \in \mathbb{R}^n$ be given. Suppose that* 1) $A \in \mathbb{R}^{n\times n}$ *is a real nonsingular matrix;* 2) $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is convex and Lipschitz continuous with Lipschitz constant $K > 0$. Then there exists $\bar{x} \in \mathbb{R}^n$ such that*

$$\langle A\bar{x} - f, v - \bar{x}\rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in \mathbb{R}^n.$$

Proof: The result is a consequence of Theorem 3.2 with Λ defined by

$$\Lambda(x) = Ax - f, \quad \forall x \in \mathbb{R}^n.$$

The matrix A is nonsingular and thus there exists $c > 0$ such that $\|Ax\| \geq c\|x\|, \quad \forall x \in \mathbb{R}^n$. Let us choose

$$r > \max\left\{\frac{K + \|f\|}{c}, \|A^{-1}f\|\right\}.$$

We see that if $\|x\| = r$, then

$$\|\Lambda(x)\| \geq \|Ax\| - \|f\| \geq c\|x\| - \|f\| > K.$$

On the other hand, we remark that

$$h(\lambda, x) := Ax - \lambda f \neq 0, \quad \forall \lambda \in [0,1], x \in \partial\mathbb{B}_r.$$

Indeed, suppose on the contrary that there exists $\lambda \in [0,1]$ and $x \in \partial\mathbb{B}_r$ such that $Ax = \lambda f$. Then

$$\|x\| = \lambda\|A^{-1}f\| \leq \|A^{-1}f\|$$

and we obtain the contradiction $r \leq \|A^{-1}f\|$. Thus

$$\begin{aligned}\deg(A. - f, \mathbb{B}_r, 0) &= \deg(h(1,.), \mathbb{B}_r, 0)\\ &= \deg(h(0,.), \mathbb{B}_r, 0)\\ &= \deg(A., \mathbb{B}_r, 0)\\ &= \text{ sign } (\det A)\\ &\neq 0,\end{aligned}$$

which completes the proof. □

4 The Poincaré operator

Let us first recall some general existence and uniqueness result (see e.g. [18]).

Theorem 4.1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous operator such that for some $\omega \in \mathbb{R}$, $F + \omega I$ is monotone,* i.e.,

$$\langle F(x) - F(y), x - y\rangle \geq -\omega\|x - y\|^2, \quad \forall x,\ y \in \mathbb{R}^n.$$

Suppose that $f : [0, +\infty) \to \mathbb{R}^n$ satisfies

$$f \in C^0([0, +\infty); \mathbb{R}^n),\ \frac{df}{dt} \in L^1_{\text{loc}}(0, +\infty; \mathbb{R}^n).$$

Let $u_0 \in \mathbb{R}^n$ and $0 < T < +\infty$ be given. There exists a unique $u \in C^0([0, T]; \mathbb{R}^n)$ such that

$$\tfrac{du}{dt} \in L^\infty(0, T; \mathbb{R}^n); \tag{13}$$

$$u \text{ is right-differentiable on } [0, T); \tag{14}$$

$$u(0) = u_0; \tag{15}$$

$$\left\langle \tfrac{du}{dt}(t) + F(u(t)) - f(t), v - u(t)\right\rangle + \varphi(v) - \varphi(u(t)) \geq 0,$$
$$\forall v \in \mathbb{R}^n,\ a.e.\ t \in [0, T]. \tag{16}$$

Remark 4.1. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is of the type

$$F(x) = Ax + \Psi'(x) + F_1(x), \quad \forall x \in \mathbb{R}^n,$$

where $A \in \mathbb{R}^{n\times n}$ is a real matrix, $\Psi \in C^1(\mathbb{R}^n; \mathbb{R})$ is convex and F_1 is Lipschitz continuous, i.e.,

$$\|F_1(x) - F_1(y)\| \leq k\|x - y\|, \quad \forall x, y \in \mathbb{R}^n,$$

for some constant $k > 0$. Then F is continuous and $F + \omega I$ is monotone provided that

$$\omega \geq \sup_{\|x\|=1} \langle -Ax, x\rangle + k.$$

We note that if F is k-Lipschitz, then $F + kI$ is monotone.

Remark 4.2. i) The variational inequality in (16) can also be written as the differential inclusion

$$\frac{du}{dt}(t) + F(u(t)) - f(t) \in -\partial\varphi(u(t)), \text{ a.e. } t \in [0, T], \tag{17}$$

ii) Let $u : [0, T] \to \mathbb{R}$ be the unique solution of (13)-(16). Then

$$\left\langle \frac{du}{dt}(t) + F(u(t)) - f(t), \xi\right\rangle + \varphi'(u(t); \xi) \geq 0, \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } t \in [0, T].$$