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# Global Solutions of Nonlinear Schrödinger Equations

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# Global Solutions of Nonlinear Schrödinger Equations

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ABSTRACT. The aim of this book is to describe recent progress on various issues in the theory of nonlinear dispersive equations, primarily the nonlinear Schrödinger equation (NLS). In particular, the Cauchy problem for the defocusing critical NLS with radial data is discussed. New techniques and results are described on global existence of large data solutions below the energy norm. Current research in Harmonic Analysis around Strichartz' inequalities and its relevance to nonlinear PDE is presented. Also several topics in NLS theory on bounded domains are reviewed. In this respect, a partial survey is given of the theory of invariant Gibbs measures and recent developments in KAM theory for PDE's.

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# Global Solutions of Nonlinear Schrödinger Equations

# Contents

## 0. Introduction and summary

Despite the attention this theory has received over recent years, there are many problems left essentially unsolved concerning the longtime behaviour of solutions to the Cauchy problem for the nonlinear Schrödinger equation (NLS for short)

$$\begin{cases} iu_t + \Delta u \pm u|u|^{p-2} = 0 \\ u(0) = \phi \in H^s(\mathbb{R}^d) \end{cases} \tag{0.1}$$

with Hamiltonian

$$H(\phi) = \int_{\mathbb{R}^d} \left[ \frac{1}{2} |\nabla \phi|^2 \mp \frac{1}{p} |\phi|^p \right] dx. \tag{0.2}$$

Although the initial value problem (IVP) theory is satisfactory for local time behaviour and small data, many issues on the behaviour of solutions for large data are far from understood. In the following case $\big($i.e. "+" sign in (0.1)$\big)$, it is well known that for $p \geq 2 + \frac{4}{d}$, smooth solutions of (0.1) may blowup in finite time. There is a vast problematic in this context, concerning questions such as blowup speed, blowup profile and its stability etc., pursued both purely mathematically and numerically. This body of problems will not be our primarily issue here and we will only comment on a few aspects of recent research. We will rather concentrate on equation $(0,1)$ in the defocusing case, when it is expected that local solutions extend to global ones and preserve their $H^s$-class for all time, with scattering behaviour for sufficiently high degree nonlinearity. We are particularly interested in two problems that we describe briefly next.

### (i) The $H^1$-critical equation

Consider the NLS

$$iu_t + \Delta u - u|u|^{p-2} = 0, \; p = 2 + \frac{4}{d-2} \qquad (d \geq 3) \tag{0.3}$$

for which the homogeneous $H^1$-space $\dot{H}^1$ is the scale invariant Sobolev space. It is known that there is local wellposedness for any data $\phi \in H^s$, $s \geq 1$ and the result is global for data small in $\dot{H}^1$. It is an open problem whether classical solutions exist global in time. Remark that since the Hamiltonian (and the $L^2$-norm) provide the only apriori bounds on the solution, also a classical theory needs to include a considerable component that is purely $H^1$. We have solved the question for radial data for $d = 3, 4$, proving global wellposedness and scattering in the energy space and any $H^s$, $s > 1$. The corresponding result for the nonlinear wave equation (NLW)

$$\Box y + y^{p-1} = y_{tt} - \Delta y + y^{p-1} = 0 \tag{0.4}$$

was established some time ago by Struwe [Str] in the radial case and by Grillakis [Gr] in general; see also the paper [S-S]. The main problem in the NLS-case is that the corresponding Morawetz-type inequality is apriori to weak to exclude $H^1$-concentration phenomena. This is the main issue in this questions.

The results for NLS appear in [B1]. In 3D, the proof presented here is a bit less technical we believe. The problem in 4D (and higher dimension) comes from the lower degree nonlinearity (the quintic nonlinearity is exploited in the 3D proof). The method followed here for the equation

$$iu_t + \Delta u - u|u|^2 = 0 \tag{0.5}$$

in 4D compared with the presentation in [B1] is less dependent on the particular nonlinearity.

For general (non-radial) data, the problem of global wellposedness is still open, also for classical solutions.

(ii) **Wellposedness below the energy norm**

The IVP

$$\begin{cases} iu_t + \Delta u \pm u|u|^{p-2} = 0 \\ u(0) = \phi \in H^s \end{cases} \tag{0.6}$$

is locally wellposed if we assume

$$s \geq 0$$

and if $p > 2 + \frac{4}{d}$

$$s \geq s_*, \; s_* \text{ defined by } p = 2 + \frac{4}{d - 2s_*}. \tag{0.7}$$

Moreover, if $s > s_*$, the time interval $\Delta T$ may be bounded below by a function of $\|\phi\|_{H^s}$. It follows that in the defocusing case there is global wellposedness in the energy space provided $p < 2 + \frac{4}{d-2}$. Our interest here is to get global results below the energy-norm. An optimal result would be to show that in the defocusing case, under assumptions (0.7), the local solution of (0.6) extends to a global one. This is unknown for large data, even in the $L^2$-critical case

$$p = 2 + \frac{4}{d} \tag{0.8}$$

(the conformal equation).

We did however develop a new and rather general method to obtain global wellposedness results for data $\phi \in H^s$, for certain $s < 1$. This method exploits the apriori bound on the $H^1$-norm from the Hamiltonian conservation, although the data is below that threshold. It is based on decomposing in a suitable way the solution in its low and high Fourier modes. As an example, the following fact is established in [B2].

The IVP in 2D

$$\begin{cases} iu_t + \Delta u - u|u|^2 = 0 \\ u(0) = \phi \in H^s(\mathbb{R}^2) \end{cases} \tag{0.9}$$

is globally wellposed provided $s > \frac{3}{5}$ and moreover

$$u(t) - e^{it\Delta}\phi \in H^1 \text{ for all time .} \tag{0.10}$$

As mentioned, our technique has general features and is not restricted to NLS. We have investigated also certain examples of nonlinear wave equations

$$\begin{cases} \Box y + \partial_y f(x,y) = y_{tt} - \Delta y + \partial_y f(x,y) = 0 \\ \big(y(0), \dot{y}(0)\big) \in H^s \times H^{s-1}. \end{cases} \tag{0.11}$$

The case $s = \frac{1}{2}$ is of particular interest since this corresponds to the symplectic space. Establishing a global flow on the symplectic Hilbert space is certainly of interest in view of applying the symplectic capacity theory (considering say periodic boundary conditions) as developed by many authors starting from Gromov's work (our reference will be [Kuk1] for the theory in infinite symplectic dimensional symplectic phase space).

Global wellposedness in the symplectic space is proven in particular for the NLW

$$y_{tt} - \Delta y + \rho y + y^3 = 0 \tag{0.12}$$

with periodic bc in $D = 1, D = 2$. These results are also new.

The first chapter of the paper is more of a survey type. We also indicate some results on derivative NLS of the form

$$iu_t + \Delta u + F(u, \overline{u}, \nabla_x u, \nabla_x \overline{u}) = 0 \tag{0.13}$$

(cf. [K-P-V] and subsequent papers). This topic is again a most interesting issue that will not be considered here. In fact, it is fair to say that most of the theory around (0.13) deals with local in time results, except for small data.

In chapter II, we will comment on a few related directions of current research that will not be developed further here. These include

(1) Perturbations of the groundstate solution for the conformal NLS in the focusing case and applications to blowup solutions.
(2) Fourier restriction theory beyond $L^2$; relation to problems of combinatorial type such as the dimension conjecture for Besicovitch sets; applications to the maximal function associated to the linear Schrödinger group and to $L^2$-concentration phenomena for NLS.
(3) Further results on derivative NLS.

In chapter III, we discuss the defocusing $H^1$-critical NLS (0.3) in the radial case.

In Chapter IV, we consider the problem of establishing global solutions below the energy norm for defocusing $H^1$-subcritical NLS and NLW.

In Chapter V of this paper, we survey investigations related to NLS on bounded spatial domains, mainly the case of periodic b.c. The problems here are different from the $\mathbb{R}^d$-case, partly because of the absence of dispersion. Besides the Cauchy problem, we will discuss results and problems related to invariant Gibbs measures and the existence and persistency of invariant KAM (Kolmogorov-Arnold-Moser) tori. Again all these topics are active research areas.

There are two Appendices included.

Appendix 1 deals with the problem of growth of higher Sobolev norms in linear Schrödinger equations with bounded, smooth, time periodic potential $V = V(x,t)$, thus of the form

$$iu_t + \Delta u + V(x,t)u = 0 \qquad u(0) = \phi \in H^s \tag{0.14}$$

(periodic bc). Although in the nonlinear context, this problem is far from understood, for equation (0.14) a very satisfactory and surprisingly general result may be shown (in any dimension)

$$\|u(t)\|_{H^s} < C_\varepsilon |t|^\varepsilon \ \|\phi\|_{H^s} \text{ when } |t| \to \infty, \text{ for all } \varepsilon > 0. \tag{1.15}$$

Observe that there is no specified behaviour of $V$ in time $t$, besides smoothness. We consider the $D = 1$ case. See [B14] for general dimension and further results.

In Appendix 2, we will summarize research over the recent years on the Zakharov system

$$\begin{cases} iu_t = -\Delta u \pm nu \\ n_{tt} - c^2 \Delta n = c^2 \Delta(|u|^2) \end{cases} \tag{0.16}$$

(the physical meaning of $u, n, c$ are respectively the electrostatic envelope field, the ion density fluctuation field and the ion sound speed). The cubic NLS

$$iu_t + \Delta u \pm u|u|^2 = 0 \tag{0.17}$$

may thus be viewed as the limit of (0.16) when $c \to \infty$.

Global existence of classical solutions for the defocusing 3D equation was only proven recently (in joint work with J. Colliander cf. [B-C]). Considering periodic bc, we will also discuss the invariant measure problem in 1D.

The present Notes are based on AMS Colloquium Lectures given in Cincinatti (1994), lectures given at Park City in 1995 and UCLA 1998. Part of the material is not published elsewhere.

# I. An overview of results on the Cauchy problem for NLS

## 1. Equations

A first distinction should be made between equations without (resp with) a presence of derivatives in the nonlinearity. Thus

$$iu_t + \Delta u + F(u, \overline{u}) = 0 \qquad \text{(without derivatives)} \tag{1.1}$$

$$iu_t + \Delta u + F(u, \overline{u}, \nabla_x u, \nabla_x \overline{u}) = 0 (F \text{ involving first order derivatives}). \tag{1.2}$$

In this chapter, our spatial domain will be mainly $\mathbb{R}^d, d = 1, 2, 3$. The case of bounded domains, say periodic boundary conditions ($x \in \mathbb{T}^d = d$ dimensional torus) will be more the subject of Chapter V.

The Cauchy problem for (1.1) has been extensively studied and sharp results obtained, especially in the case

$$F(u, \overline{u}) = \frac{\partial}{\partial \overline{u}}(|u|^p) \sim |u|^{p-2}u. \tag{1.3}$$

The equation

$$iu_t + \Delta u + \frac{\partial H_0}{\partial \overline{u}}(u, \overline{u}) = 0 = iu_t + \frac{\partial H}{\partial \overline{u}}$$

is Hamiltonian, with Hamiltonian

$$H(\phi) = \frac{1}{2}\int |\nabla\phi|^2 - \int H_0(\phi), \; H_0(\phi) = |\phi|^p \tag{1.4}$$

preserved under the flow.

The "natural" symplectic Hilbert space is the space $L^2$ with canonical coordinates (formally) ($\operatorname{Re} u$, $\operatorname{Im} u$).

In the case (1.3) or more generally

$$H_0 = H_0(|u|^2)$$

there is also conservation of the $L^2$-norm

$$\left(\int |\phi|^2\right)^{1/2}$$

under the flow.

In case (1.3), i.e.

$$iu_t + \Delta u + \lambda u|u|^{p-2} = 0 \tag{1.5}$$

with Hamiltonian

$$\frac{1}{2}\int |\nabla u|^2 - \frac{\lambda}{p}\int |u|^p \tag{1.6}$$

we distinguish the cases

$$\lambda > 0 \quad = \text{focusing case}$$
$$\lambda < 0 \quad = \text{defocusing case}$$

In the defocusing case (and $p \leq 6$ for $d = 3$), the conservation of (1.6) apriori bound on $\|u(t)\|_{H^1}$ for $\phi \in H^1$.

The case

$$p = 2 + \frac{4}{d}$$

is special and called the pseudo-conformally invariant case. If $u$ is a solutioı so is

$$Cu(x,t) = \frac{1}{|t|^{d/2}} e^{i\frac{|x|^2}{4t}} u\Big(\frac{x}{t}, \frac{-1}{t}\Big)$$

which gives an additional symmetry.

Based on the pc-transformation, one gets for $p = 2 + \frac{4}{d}$ the pseudo-con conservation law

$$\|(x + 2it\nabla)u(t)\|_2^2 - \frac{8\lambda t^2}{p}\|u(t)\|_p^p = \|x\varphi\|_2^2.$$

In the general case, there is an additional term

$$\|(x + 2it\nabla)u(t)\|_2^2 - \frac{8\lambda t^2}{p}\|u(t)\|_p^p =$$

$$\||x|\varphi\|_2^2 - \frac{4\lambda}{p}(4 - d(p-2)) \int_0^t s \int_{\mathbb{R}^d} |u(s,x)|^p dx\, ds.$$

This pc conservation law is particularly useful in the defocusing case $\lambda < 0$, $p \geq 2 + \frac{4}{d}$ and $|x|\varphi \in L^2$, since one gets the apriori bound

$$\|u(t)\|_p^p < \frac{C}{t^2}.$$

## 2. Wellposedness of the Cauchy problem

We consider the case

$$iu_t + \Delta u + \lambda u|u|^{p-2} = 0.$$

Most of the results stated below have extensions to more general situations. tion (2.7) is invariant under the scaling

$$u(x,t) \to a^{\frac{2}{p-2}} u(ax, a^2 t).$$

Hence, putting

$$p - 2 = \frac{4}{d - 2s_0}$$

($s_0$ = scaling exponent)

the (homogeneous) Sobolev space $H^{s_0}$ is invariant under the scaling (2.1).

We then distinguish the cases $s = s_0, s > s_0$

$$s = s_0 \ : \quad \text{critical}$$
$$s > s_0 \ : \quad \text{subcritical}$$

HEOREM 1. *(local wellposedness).*

*ssume $u(0) = \varphi \in H^s$, $s \geq 0$ and $s \geq s_0$. Assume also $p - 2 > [s]$ if $p \notin 2\mathbb{Z}$. the Cauchy problem*

$$\begin{cases} iu_t + \Delta u + \lambda u|u|^{p-2} = 0 \\ u(0) = \phi \in H^s \end{cases}$$

*'posed on a nontrivial time interval $[0, T^*[$ and in particular*

$$u \in C_{H^s}([0, T]),\ T < T^*.$$

*subcritical case $s > s_0$, $T^* > T(s, \|\varphi\|_{H^s})$ and the flowmap is Lipschitz on a ›orhood of $\phi$.*

**emark 1.** In the critical case, maximal existence time depends on $\varphi$, not $\varphi\|_{H^s}$.

**emark 2.** Take $d = 2, s = 0, p = 2 + \frac{4}{d} = 4$ and assume $T^* < \infty$. Then

.3) $$\lim_{t \vec{<} T^*} \sup_{\substack{I \subset \mathbb{R}^d \\ |I| = (T^* - t)^{1/2}}} \int_I |u(x,t)|^2 dx > c \qquad (c = \text{ fixed constant})$$

result, valid as well in the focusing as defocusing case, expresses a precise ıcentration phenomenon of $L^2$-norm).

is not known if this result is optimal however.

ext result deals with the question when the local solution provided by Theo- extends to a global one.

HEOREM 2. *(global solutions)*

*'he local solution given by Theorem 1 extends to a global one in the following*

$p < 2 + \frac{4}{d}$

*(problem $L^2$-subcritical and use of $L^2$-conservation)*

$p \geq 2 + \frac{4}{d}$ *and small $H^s$-data ($s \geq s_0$)*

*Defocusing case, $p < 6$ for $d = 3$ and $\varphi \in H^1$*

*(problem $H^1$-subcritical and use of Hamiltonian conservation)*

*Also true for $d = 3, p = 6$ and $\varphi \in H^1$ a radial function, i.e. $H^1$-critical case (recent result discussed in Chapter III)*

*Defocusing case, $p < 6$ for $d = 3$, $\varphi \in H^s (s \geq s_0)$ and $|x|\varphi \in L^2$*

*(use of apriori bound on $\|u(t)\|_p$ from pc conservation law (1.10))*

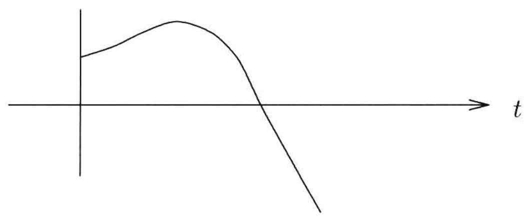
*foreover, additional smoothness of data $\varphi$ is preserved under the flow (provided ıtible with smoothness of nonlinearity (see remarks below)).*

**Remark 1.** For $p \geq 2 + \frac{4}{d}$ in the focusing case, (sufficiently large) smooth solutions may blowup in finite time

(i) **Glassey's viriel inequality**

$$\frac{d^2}{dt^2}\left[\int |\varphi|^2 |x|^2 dx\right] \leq cH(\phi) \qquad (c > 0). \tag{2.4}$$

Hence, if $H(\phi) < 0$, blowup has to occur




(ii) For $p = 2 + \frac{4}{d}$, there are constructions of explicit blowup solutions (of minimal $L^2$-norm) from groundstate and pc transformation.

THEOREM 3. *(F. Merle, [M1])*

*Let $u$ be a solution of*

$$\begin{cases} iu_t + \Delta u + u|u|^{4/d} = 0 \\ u(0) = \phi \end{cases}$$

*where $\phi \in H^1$ and*

$$\|\phi\|_2 = \|Q\|_2$$

*where $Q$ denotes groundstate, i.e. (unique) solution of*

$$\Delta Q + Q^{1+4/d} = Q, Q \text{ positive and radial.} \tag{2.5}$$

*Assume $u$ blows up at time $T > 0$.*

*Then there exist $\theta \in \mathbb{R}, \omega > 0, x_0 \in \mathbb{R}^d, x_1 \in \mathbb{R}^d$ such that*

$$u(t,x) = \left(\frac{\omega}{T-t}\right)^{d/2} e^{i\{\theta + (t-T)^{-1}\left[\frac{|x-x_1|^2}{4} - \omega^2\right]\}} Q\left(\frac{\omega(x-x_1)}{T-t} - \omega x_0\right) \tag{2.6}$$

*(Uniqueness of minimum $L^2$-norm blowup solutions).*

THEOREM 4. *[M2]*

*Given a time $T$ and distinct points $x_1, \dots, x_K \in \mathbb{R}^d$, there is a solution $u$ of*

$$iu_t + \Delta u + u|u|^{4/d} = 0$$

*which blows up exactly at time $t = T$ in the points $\{x_1, \dots, x_K\}$ with concentration of all the $L^2$-mass on this finite set of points.*

THEOREM 5. *[B-W]*

*Let $d = 1, 2$ (for smoothness reasons).*

*Denote $u_0$ the explicit blowup solution at $(T, x_1)$ of $iu_t + \Delta u + u|u|^{4/d} = 0$ given by (2.6). Then one may construct solutions $u = u_0 + v$ on $[0, T[$ where $v$ is smooth ($\neq 0$) extending smoothly after blowup time $T$ and solving for $T < t < T + 1$ the IVP*

$$\begin{cases} iv_t + \Delta v + v|v|^{4/d} = 0 \\ v(T) = \phi. \end{cases}$$

*Here $\phi$ may be taken any sufficiently small smooth function on $\mathbb{R}^d$, vanishing at sufficiently high order at $x = x_1$ and of fast decay at infinity.*

The preceding result shows in particular that at blowup time all of the $L^2$-norm need not be absorbed in the blowup. Theorem 5 is a more recent result and we will comment on it in the next chapter.

**Remark 2.** Assume $\varphi \in H^s$, $s > 1$ and $p < 6$ for $d = 3$ and nonlinearity sufficiently smooth

(i) If $\sup \|u(t)\|_{H^1} < \infty$ (in particular in the defocusing case), then $u(t) \in H^s$ for all time and

$$\|u(t)\|_{H^s} < (1 + |t|)^{C(s-1)}.$$

For $d = 3$,

$$\|u(t)\|_{H^s_{\text{loc}}} < C(\|\varphi\|_{H^s})$$

(ii) Assume *defocusing case*, $p > 2 + \frac{4}{d}$ (and $p < 6$ for $d = 3$).

If moreover $|x|\varphi \in L^2$, then

$$\|u(t)\|_{H^s} < C(\|\varphi\|_{H^s}, \|x\varphi\|_2)$$

(and one has scattering in $H^s$).

Same statement holds for $d = 3$ without decay assumption, thus if $d = 3$, defocusing, $p < 6$, sufficiently smooth nonlinearity (depending on $s$), then

$$\varphi = u(0) \in H^s, s \geq 1 \Rightarrow \|u(t)\|_{H^s} < C(\|\varphi\|_{H^s}) \text{ for all time}$$

and one has moreover scattering in $H^s$.

(this is in particular the case for $p = 4$).

**Remark 3.** The main problems regarding the global Cauchy problem

(1) **$L^2$-critical defocusing case** ($p = 2 + \frac{4}{d}$, i.e. $L^2$=scale invariant space).

Is $T^* = \infty$ for $\varphi \in L^2$?

Known in the following cases

(i) $\|\varphi\|_2$ small
(ii) $\varphi \in H^1$
(iii) $|x|\varphi \in L^2$

For $d = 2$

Case (ii) may be improved to $\varphi \in H^s$, $s > \frac{3}{5}$

Case (iii) may be improved to $|x|^s \varphi \in L^2$, $s > \frac{3}{5}$

(2) $H^1$**-critical case** ($H^1$ = scale invariant space)

$3D$ $\quad iu_t + \Delta u - u|u|^4 = 0$

Do classical solutions exist for all time?

Recently proved affirmative in radial case

[Known (affirmatively) for the 3D wave equation (Struwe, Grillakis)

$$\Box y + y^5 = y_{tt} - \Delta y + y^5 = 0].$$

Similar problem and result in 4D for equation

$$iu_t + \Delta u - u|u|^2 = 0.$$

## 3. Scattering results

(1.5) has the following equivalent integral equation

$$\begin{aligned} u(t) &= e^{it\Delta}\varphi - i\lambda \int_0^t e^{i(t-\tau)\Delta}(u|u|^{p-2})(\tau)d\tau \\ &= e^{it\Delta}(\Omega_+\varphi) + i\lambda \int_t^\infty e^{i(t-\tau)}(u|u|^{p-2})(\tau)d\tau \end{aligned}$$

where

$$\Omega_+\varphi = \varphi - i\lambda \int_0^\infty e^{-i\tau\Delta}(u|u|^{p-2})(\tau)d\tau.$$

Scattering in $H^s$ means

$$\Omega_+\varphi \in H^s$$
$$\|u(t) - e^{it\Delta}(\Omega_+\varphi)\|_{H^s} \overset{t\to\infty}{\longrightarrow} 0.$$

We list a number of instances where scattering is known to occur.

(1) **Small data scattering**

$p \geq 2 + \frac{4}{d}$.

Essentially byproduct of IVP-analysis.

(2) **Global scattering with decay**

Assume defocusing NLS and

$$\frac{2 - d + \sqrt{d^2 + 12d + 4}}{2d} = a(d) < p - 2 (< 4 \text{ for } d = 3)$$
$$\left(\frac{2}{d} < a(d) < \frac{4}{d}\right).$$