Finite Simple Groups II

Edited by

Michael J. Collins

FINITE SIMPLE GROUPS II

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Michael J. Collins
University College, Oxford

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Preface

This book is the outcome of a Research Symposium in Finite Simple Groups which was held at the University of Durham, July 31–August 10, 1978, under the auspices of the London Mathematical Society. The dominant area of discussion was the classification of simple groups, a programme which is now almost complete. In our original proposal for the symposium, we did not feel that this should be the exclusive topic, and there were also lectures and seminars on the representation theory of simple groups of Lie type and on geometry and sporadic groups. This book is correspondingly divided into three parts.

Part A covers finite simple groups and their classification. While formally this book is a proceedings of the symposium, this material has been presented in a form which will serve as a survey of the eventual complete classification, and as an introduction to the more detailed arguments that appear in the individual research papers in the literature, both now and in the future. This is the purpose of the first eight chapters, and, with this in mind, the opportunity has been taken during the preparation for publication to update the factual information presented as much as possible. Because of their origin, the emphasis is naturally on recent work, and no details are given of much of the necessary work which preceded Gorenstein's formal programme for a classification, though the first chapter does include an historical background and a brief account of the known simple groups. The last two chapters are a little different, and are related more to a re-examination and axiomatization of the ideas and methods employed.

The core of this part of the symposium consisted of series of lectures given by M. Aschbacher, G. Glauberman and D. Gorenstein, and they have contributed their lectures, in Gorenstein's case with R. Lyons on their joint work. There were a number of seminars on various aspects of the classification and related topics; for most of these, their organisers have written unified accounts of the subjects. In addition, Gorenstein gave an opening lecture as an introduction and survey of the entire classification programme. In writing the corresponding sections of Chapter 1 (Sections 4–6), I have leaned heavily on his notes, but have considerably expanded on his account so as to link the remaining chapters, and give the necessary notation and definitions not special to a particular chapter. In addition, because of the unity of the material, I have prepared a common list of references for Part A; this should *not* be considered as a complete bibliography for the subject. Books are separated from articles; in particular, most of the basic group theory which is freely assumed can be found in the texts by Gorenstein [G] and Huppert [H], while a more specialised background can be found in Carter [C] and the Proceedings of the 1969 Oxford L.M.S. Instructional Conference [FSG].

The remainder of the book covers the other topics. R. W. Carter and J. E. Humphreys gave series of lectures on the representation theory of finite simple groups of Lie type as "surveys for group theorists". These appear in Part B, together with an account by J. Jantzen of his own work, given in the seminar on representation theory. We invited J. Tits and J. G. Thompson to give lectures. Tits gave four lectures entitled "The geometry of sporadic groups". In contributing two chapters, he has excluded his lecture on the Mathieu groups. Thompson lectured on an application of representation theory to questions about finite projective planes and his work appears here. These comprise Part C, together with a general account of his computer constructions of sporadic groups which Sims gave to the sporadic groups seminar. (This seminar also included a considerable discussion of Fischer's Monster, but this is not included; in fact, relatively little was known then as compared with now!)

It is a pleasure to acknowledge the financial support given to the Symposium by the Science Research Council, and also to thank the National Science Foundation for their partial support of many of the participants from the United States. Personally, I should like to thank Roger Carter for his advice and assistance with the programme as co-organiser, Tom Willmore and Lyndon Woodward in Durham, and Valerie Willoughby for her assistance with the organisation in Oxford. Not least, I should like to thank all those who participated in the Symposium and contributed to its success.

In the preparation of this volume, I am extremely grateful to all the contributors for their cooperation and to the Academic Press for their patience in waiting for the manuscript. I am also grateful to the California Institute of Technology where I have held a visiting appointment while preparing this volume, to David Wales for his advice and criticism while I was writing Chapter 1, and to Frances Williams and Lillian Chappelle who have typed portions of the manuscript.

Pasadena, California M. J. Collins
October 1979

Notation and Terminology

Much of the basic notation which is used in this book is standard, and the following list is not complete, but only intended for clarification. (Definitions required for only one chapter and given there are excluded.)

$\{\cdots \mid \cdots\}$	set of $\cdots$ such that $\cdots$
$A \subseteq B (B \supseteq A)$	A is a subset of B (also used without distinction for subgroups)
$A \subset B (B \supset A)$	A is a proper subset of B
$A - B$	$\{a \in A \mid a \notin B\}$

If G is a group:

$H \lhd G (H \trianglelefteq G)$	H is a normal subgroup of G
$H \lhd\lhd G (H \trianglelefteq\trianglelefteq G)$	H is a subnormal subgroup of G
$o(g)$	order of an element $g \in G$
x^g	$g^{-1}xg \quad (g, x \in G)$
A^g	$g^{-1}Ag \quad (g \in G, A \subseteq G)$
$\langle \cdots \mid \cdots \rangle$	group generated by $\{\cdots \mid \cdots\}$
$[x, y]$	$x^{-1}y^{-1}xy \quad (x, y \in G)$
$[H, K]$	$\langle [h, k] \mid h \in H, h \in K \rangle \quad (H, K \subseteq G)$
$[H, K, L]$	$[[H, K], L]$
$G' = G^{(1)}$	$[G, G]$
$G^{(n)}$	$[G^{(n-1)}, G^{(n-1)}]$ (inductively)
$G^{(\infty)}$	$\bigcap G^{(n)} \quad (n \in \mathbf{N})$
$C_G(H)$	centralizer of H in G
$N_G(H)$	normalizer of H in G
$Z(G)$	centre of G
$F(G)$	Fitting subgroup of G
$\text{И}_G(H; \pi)$	set of π-subgroups of G normalized by a subgroup H (π set of primes)
$\text{И}_G^*(H; \pi)$	maximal elements of $\text{И}_G(H; \pi)$ under inclusion

If p is a prime, p' denotes the complementary set of primes:

$O_p(G)$	largest normal p-subgroup of G
$O_{p'}(G)$	largest normal p'-subgroup of G
$O_{p',p}(G)$	subgroup of G containing $O_{p'}(G)$ such that $O_{p',p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G))$
$O(G) = O_{2'}(G)$	*core* of G
$Z^*(G)$	subgroup of G containing $O(G)$ such that $Z^*(G)/O(G) = Z(G/O(G))$
$O^p(G)$	smallest normal subgroup of G of index a power of p

$O^{p'}(G)$	smallest normal subgroup of G of index prime to p
$Syl_p(G)$	set of Sylow p-subgroups of G
If P is a p-group:	
$\Omega_1(P)$	$\langle x \in P \mid x^p = 1 \rangle$
$\mho^1(P)$	$\langle x^p \mid x \in P \rangle$
$\Phi(P)$	Frattini subgroup of P
Also:	
$F_q = GF(q)$	finite field of order q
$H \circ K$	central product of groups H, K
$H \wr K$	wreathed product of H by K
Z_n	cyclic group of order n
D_{2n}	dihedral group of order $2n$
S_n	symmetric group of degree n
A_n	alternating group of degree n
$G(q)$	group of Lie type defined over $GF(q)$
$Chev(p)$	$\{G(q) \mid p \text{ divides } q\}$ (includes groups of Ree type if $p = 3$)

See Chapter 1, Section 3, for detailed notation for the known simple groups.

The following table contains notation which is defined in Chapter 1.

Notation	Meaning	Page of definition
$m(P)$	rank of a p-group P	26
$r(P)$	sectional rank of P	27
$m_p(G)$	p-rank of G	27
$m_{2,p}(G)$	2-local p-rank of G	36
$e(G)$	odd 2-local rank of G	36
$SCN(P)$		27
$SCN_3(P)$		27
$\Gamma_{P,k}(G)$	k-generated (p-)core of G	28
$\Gamma^0_{P,2}(G)$	weak 2-generated p-core	39
$F^*(G)$	generalized Fitting subgroup of G	32
$E(G) = L(G)$	layer (extended socle) of G	32
$L_{2'}(G)$	2-layer of G	33
$B(G)$		33

The following terminology, additional to that above, is introduced in Chapter 1.

Contents

Part A

Finite Simple Groups and Their Classification

1

Introduction: A Survey of the Classification Project

MICHAEL J. COLLINS

1. Background

The current attempt to classify the finite simple groups has its origin, as a systematic programme, in Brauer's address at the International Congress in Amsterdam in 1954 [33]. Brauer had proved that if G was a finite simple group of even order and t was an involution (element of order 2) in G, then

$$|G| \leqslant (|C_G(t)|^2)!$$

Thus, if one specifies a group H having centre of even order, there can be only finitely many simple groups G which contain an involution t for which $C_G(t)$ is isomorphic to H. The bound in question is not a useful one, but its mere existence encouraged Brauer to suggest a systematic characterization of known simple groups in this way and he was able to obtain such results, using character-theoretic methods to obtain enough information to construct G from a knowledge of H in certain specific cases.

What does one mean by "classification of finite simple groups"? Results such as those obtained by Brauer are often termed *specific* characterizations, even though H may have been specified only as a member of a class of groups. Indeed it can happen, as we shall see later, that a judicious choice of a group H may result in the characterization of a previously unknown simple group. By contrast, a *general* characterization will consist of the determination of those simple groups (or suitable information about unknown groups) satisfying some general inductive property; for example, having odd order (yielding the empty set) or having abelian Sylow 2-subgroups (allowing for the then indeterminacy over the Ree groups, for which the term *group of Ree type* was introduced). Because the nature of proofs is usually inductive, it is normally the case that one characterizes *all* groups with a given property and deduces the list of simple groups as a corollary; for example, one often knows nothing about the structure of normal subgroups of odd order, and this is why the term *characterization* is used. However, the ultimate *classification* theorem will be one which says, "If G is a finite simple group, then G is isomorphic to a known group".

In practice, we may settle for a little less. In effect, the current attempt to classify finite simple groups, as will be described in this book, is a proof by induction. One should start with a minimal counterexample. However, the methods are such that one can ignore certain "indeterminacies"; all that is required is that one know those properties of a possible unknown simple group that are needed for the arguments. Then the question of actual existence or uniqueness can be left as a separate problem; at worst, resolution of this question can result in some of the previous arguments becoming superfluous.

The first ingredient needed in this process, therefore, is a list of the known simple groups. (Here, as throughout Part A, we shall suppress the word "finite" as understood.) The search for simple groups has always been a part of the subject, as much as a source of examples as in their own right. By early this century, the finite analogues of the classical groups had been discovered, plus two exceptional families found by Dickson (now known as $G_2(q)$ and $E_6(q)$, but discovered as groups leaving certain forms invariant), in addition to the alternating groups and the five Mathieu groups. The existence of the Mathieu groups was long questioned; they were found as multiply transitive permutation groups and were described by Burnside as "apparently sporadic groups" since they did not fit into any of the infinite families.

Despite the work of Burnside and Frobenius, there was an enormous gulf between the general theory of finite groups that developed around that time and the particular knowledge of the simple groups then known; in particular, and highly relevant to modern study, the conjectures of Frobenius and Burnside on the nilpotence of the kernel of a Frobenius group and the solubility of groups of odd order, respectively, were, as far as we know, completely untouched. In both cases, a minimal counterexample is simple. For Burnside's conjecture, this is trivial; in the case of Frobenius' conjecture, one first needs to know that the conjecture holds for soluble groups, and this fact is usually ascribed to Witt.

While progress in group theory continued, most notably p-groups, soluble groups, permutation groups and representation theory, the specific study of simple groups remained largely dormant for nearly fifty years. Then three major developments regenerated interest in the subject.

The first was the application of character theory. Brauer's project has