Graduate Texts in Mathematics

Reinhard Diestel

Graph Theory

Third Edition

Springer

Reinhard Diestel

Graph Theory

The third edition of this standard textbook of modern graph theory has been carefully revised, updated, and substantially extended.

Covering all its major recent developments it can be used both as a reliable textbook for an introductory course and as a graduate text: on each topic it covers all the basic material in full detail, and adds one or two deeper results (again with detailed proofs) to illustrate the more advanced methods of that field.

From the reviews of the first two editions (1997, 2000):

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Bulletin of the Institute of Combinatorics and its Applications

"A highlight of the book is what is by far the best account in print of the Seymour-Robertson theory of graph minors." *Mathematika*

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Third Edition

With 146 Illustrations

Springer

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To Dagmar

Preface

Almost two decades have passed since the appearance of those graph theory texts that still set the agenda for most introductory courses taught today. The canon created by those books has helped to identify some main fields of study and research, and will doubtless continue to influence the development of the discipline for some time to come.

Yet much has happened in those 20 years, in graph theory no less than elsewhere: deep new theorems have been found, seemingly disparate methods and results have become interrelated, entire new branches have arisen. To name just a few such developments, one may think of how the new notion of list colouring has bridged the gulf between invariants such as average degree and chromatic number, how probabilistic methods and the regularity lemma have pervaded extremal graph theory and Ramsey theory, or how the entirely new field of graph minors and tree-decompositions has brought standard methods of surface topology to bear on long-standing algorithmic graph problems.

Clearly, then, the time has come for a reappraisal: *what are, today, the essential areas, methods and results that should form the centre of an introductory graph theory course aiming to equip its audience for the most likely developments ahead?*

I have tried in this book to offer material for such a course. In view of the increasing complexity and maturity of the subject, I have broken with the tradition of attempting to cover both theory and applications: this book offers an introduction to the theory of graphs as part of (pure) mathematics; it contains neither explicit algorithms nor 'real world' applications. My hope is that the potential for depth gained by this restriction in scope will serve students of computer science as much as their peers in mathematics: assuming that they prefer algorithms but will benefit from an encounter with pure mathematics of *some* kind, it seems an ideal opportunity to look for this close to where their heart lies!

In the selection and presentation of material, I have tried to accommodate two conflicting goals. On the one hand, I believe that an introductory text should be lean and concentrate on the essential, so as to offer guidance to those new to the field. As a graduate text, moreover, it should get to the heart of the matter quickly: after all, the idea is to convey at least an impression of the depth and methods of the subject. On the other hand, it has been my particular concern to write with sufficient detail to make the text enjoyable and easy to read: guiding questions and ideas will be discussed explicitly, and all proofs presented will be rigorous and complete.

A typical chapter, therefore, begins with a brief discussion of what are the guiding questions in the area it covers, continues with a succinct account of its classic results (often with simplified proofs), and then presents one or two deeper theorems that bring out the full flavour of that area. The proofs of these latter results are typically preceded by (or interspersed with) an informal account of their main ideas, but are then presented formally at the same level of detail as their simpler counterparts. I soon noticed that, as a consequence, some of those proofs came out rather longer in print than seemed fair to their often beautifully simple conception. I would hope, however, that even for the professional reader the relatively detailed account of those proofs will at least help to minimize reading time...

If desired, this text can be used for a lecture course with little or no further preparation. The simplest way to do this would be to follow the order of presentation, chapter by chapter: apart from two clearly marked exceptions, any results used in the proof of others precede them in the text.

Alternatively, a lecturer may wish to divide the material into an easy basic course for one semester, and a more challenging follow-up course for another. To help with the preparation of courses deviating from the order of presentation, I have listed in the margin next to each proof the reference numbers of those results that are used in that proof. These references are given in round brackets: for example, a reference (4.1.2) in the margin next to the proof of Theorem 4.3.2 indicates that Lemma 4.1.2 will be used in this proof. Correspondingly, in the margin next to Lemma 4.1.2 there is a reference [4.3.2] (in square brackets) informing the reader that this lemma will be used in the proof of Theorem 4.3.2. Note that this system applies between different sections only (of the same or of different chapters): the sections themselves are written as units and best read in their order of presentation.

The mathematical prerequisites for this book, as for most graph theory texts, are minimal: a first grounding in linear algebra is assumed for Chapter 1.9 and once in Chapter 5.5, some basic topological concepts about the Euclidean plane and 3-space are used in Chapter 4, and a previous first encounter with elementary probability will help with Chapter 11. (Even here, all that is assumed formally is the knowledge of basic definitions: the few probabilistic tools used are developed in the text.) There are two areas of graph theory which I find both fascinating and important, especially from the perspective of pure mathematics adopted here, but which are not covered in this book: these are algebraic graph theory and infinite graphs.

At the end of each chapter, there is a section with exercises and another with bibliographical and historical notes. Many of the exercises were chosen to complement the main narrative of the text: they illustrate new concepts, show how a new invariant relates to earlier ones, or indicate ways in which a result stated in the text is best possible. Particularly easy exercises are identified by the superscript $^{-}$, the more challenging ones carry a $^{+}$. The notes are intended to guide the reader on to further reading, in particular to any monographs or survey articles on the theme of that chapter. They also offer some historical and other remarks on the material presented in the text.

Ends of proofs are marked by the symbol □. Where this symbol is found directly below a formal assertion, it means that the proof should be clear after what has been said—a claim waiting to be verified! There are also some deeper theorems which are stated, without proof, as background information: these can be identified by the absence of both proof and □.

Almost every book contains errors, and this one will hardly be an exception. I shall try to post on the Web any corrections that become necessary. The relevant site may change in time, but will always be accessible via the following two addresses:

http://www.springer-ny.com/supplements/diestel/
http://www.springer.de/catalog/html-files/deutsch/math/3540609180.html

Please let me know about any errors you find.

Little in a textbook is truly original: even the style of writing and of presentation will invariably be influenced by examples. The book that no doubt influenced me most is the classic GTM graph theory text by Bollobás: it was in the course recorded by this text that I learnt my first graph theory as a student. Anyone who knows this book well will feel its influence here, despite all differences in contents and presentation.

I should like to thank all who gave so generously of their time, knowledge and advice in connection with this book. I have benefited particularly from the help of N. Alon, G. Brightwell, R. Gillett, R. Halin, M. Hintz, A. Huck, I. Leader, T. Łuczak, W. Mader, V. Rödl, A.D. Scott, P.D. Seymour, G. Simonyi, M. Škoviera, R. Thomas, C. Thomassen and P. Valtr. I am particularly grateful also to Tommy R. Jensen, who taught me much about colouring and all I know about k-flows, and who invested immense amounts of diligence and energy in his proofreading of the preliminary German version of this book.

March 1997 *RD*

About the second edition

Naturally, I am delighted at having to write this addendum so soon after this book came out in the summer of 1997. It is particularly gratifying to hear that people are gradually adopting it not only for their personal use but more and more also as a course text; this, after all, was my aim when I wrote it, and my excuse for agonizing more over presentation than I might otherwise have done.

There are two major changes. The last chapter on graph minors now gives a complete proof of one of the major results of the Robertson-Seymour theory, their theorem that excluding a graph as a minor bounds the tree-width if and only if that graph is planar. This short proof did not exist when I wrote the first edition, which is why I then included a short proof of the next best thing, the analogous result for path-width. That theorem has now been dropped from Chapter 12. Another addition in this chapter is that the tree-width duality theorem, Theorem 12.3.9, now comes with a (short) proof too.

The second major change is the addition of a complete set of hints for the exercises. These are largely Tommy Jensen's work, and I am grateful for the time he donated to this project. The aim of these hints is to help those who use the book to study graph theory on their own, but *not* to spoil the fun. The exercises, including hints, continue to be intended for classroom use.

Apart from these two changes, there are a few additions. The most noticable of these are the formal introduction of depth-first search trees in Section 1.5 (which has led to some simplifications in later proofs) and an ingenious new proof of Menger's theorem due to Böhme, Göring and Harant (which has not otherwise been published).

Finally, there is a host of small simplifications and clarifications of arguments that I noticed as I taught from the book, or which were pointed out to me by others. To all these I offer my special thanks.

The Web site for the book has followed me to

http://www.math.uni-hamburg.de/home/diestel/books/graph.theory/

I expect this address to be stable for some time.

Once more, my thanks go to all who contributed to this second edition by commenting on the first—and I look forward to further comments!

December 1999 *RD*

About the third edition

There is no denying that this book has grown. Is it still as 'lean and concentrating on the essential' as I said it should be when I wrote the preface to the first edition, now almost eight years ago?

I believe that it is, perhaps now more than ever. So why the increase in volume? Part of the answer is that I have continued to pursue the original dual aim of offering two different things between one pair of covers:

- a reliable first introduction to graph theory that can be used either for personal study or as a course text;
- a graduate text that offers some depth in selected areas.

For each of these aims, some material has been added. Some of this covers new topics, which can be included or skipped as desired. An example at the introductory level is the new section on packing and covering with the Erdős-Pósa theorem, or the inclusion of the stable marriage theorem in the matching chapter. An example at the graduate level is the Robertson-Seymour structure theorem for graphs without a given minor: a result that takes a few lines to state, but one which is increasingly relied on in the literature, so that an easily accessible reference seems desirable. Another addition, also in the chapter on graph minors, is a new proof of the 'Kuratowski theorem for higher surfaces'—a proof which illustrates the interplay between graph minor theory and surface topology better than was previously possible. The proof is complemented by an appendix on surfaces, which supplies the required background and also sheds some more light on the proof of the graph minor theorem.

Changes that affect previously existing material are rare, except for countless local improvements intended to consolidate and polish rather than change. I am aware that, as this book is increasingly adopted as a course text, there is a certain desire for stability. Many of these local improvements are the result of generous feedback I got from colleagues using the book in this way, and I am very grateful for their help and advice.

There are also some local additions. Most of these developed from my own notes, pencilled in the margin as I prepared to teach from the book. They typically complement an important but technical proof, when I felt that its essential ideas might get overlooked in the formal write-up. For example, the proof of the Erdős-Stone theorem now has an informal post-mortem that looks at how exactly the regularity lemma comes to be applied in it. Unlike the formal proof, the discussion starts out from the main idea, and finally arrives at how the parameters to be declared at the start of the formal proof must be specified. Similarly, there is now a discussion pointing to some ideas in the proof of the perfect graph theorem. However, in all these cases the formal proofs have been left essentially untouched.

The only substantial change to existing material is that the old Theorem 8.1.1 (that cr^2n edges force a TK^r) seems to have lost its nice (and long) proof. Previously, this proof had served as a welcome opportunity to explain some methods in sparse extremal graph theory. These methods have migrated to the connectivity chapter, where they now live under the roof of the new proof by Thomas and Wollan that $8kn$ edges make a $2k$-connected graph k-linked. So they are still there, leaner than ever before, and just presenting themselves under a new guise. As a consequence of this change, the two earlier chapters on dense and sparse extremal graph theory could be reunited, to form a new chapter appropriately named as *Extremal Graph Theory*.

Finally, there is an entirely new chapter, on infinite graphs. When graph theory first emerged as a mathematical discipline, finite and infinite graphs were usually treated on a par. This has changed in recent years, which I see as a regrettable loss: infinite graphs continue to provide a natural and frequently used bridge to other fields of mathematics, and they hold some special fascination of their own. One aspect of this is that proofs often have to be more constructive and algorithmic in nature than their finite counterparts. The infinite version of Menger's theorem in Section 8.4 is a typical example: it offers algorithmic insights into connectivity problems in networks that are invisible to the slick inductive proofs of the finite theorem given in Chapter 3.3.

Once more, my thanks go to all the readers and colleagues whose comments helped to improve the book. I am particularly grateful to Imre Leader for his judicious comments on the whole of the infinite chapter; to my graph theory seminar, in particular to Lilian Matthiesen and Philipp Sprüssel, for giving the chapter a test run and solving all its exercises (of which eighty survived their scrutiny); to Angelos Georgakopoulos for much proofreading elsewhere; to Melanie Win Myint for recompiling the index and extending it substantially; and to Tim Stelldinger for nursing the whale on page 366 until it was strong enough to carry its baby dinosaur.

May 2005 *RD*

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* Sections marked by an asterisk are recommended for a first course.
Of sections marked $^{(*)}$, the beginning is recommended for a first course.