STUDIES IN MATHEMATICS AND ITS APPLICATIONS

29

MATHEMATICAL ELASTICITY

VOLUME III: THEORY OF SHELLS

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NORTH-HOLLAND

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Shell theory attempts the impossible: to provide a two-dimensional representation of an intrinsically three-dimensional phenomenon.

Warner KOITER and James SIMMONDS.

Foundations of shell theory, in *Proceedings of the Thirteenth International Congress of Theoretical and Applied Mechanics* (Moscow, 1972).

MATHEMATICAL ELASTICITY: GENERAL PLAN

VOLUME I: THREE-DIMENSIONAL ELASTICITY

Part A. Description of three-dimensional elasticity

Part B. Mathematical methods in three-dimensional elasticity

VOLUME II: THEORY OF PLATES

Part A. Linear plate theory

Part B. Nonlinear plate theory

VOLUME III: THEORY OF SHELLS

MATHEMATICAL ELASTICITY: GENERAL PREFACE [1]

This treatise, which comprises three volumes, is intended to be both a *thorough introduction to contemporary research in elasticity* and a *working textbook at the graduate level for courses in pure or applied mathematics or in continuum mechanics.*

During the past decades, elasticity has become the object of a considerable renewed interest, both in its physical foundations and in its mathematical theory. One reason behind this recent attention is that it has been increasingly acknowledged that the classical *linear equations* of elasticity, whose mathematical theory is now firmly established, have a limited range of applicability, outside of which they should be replaced by the genuine *nonlinear equations* that they in effect approximate.

Another reason, similar in its principle, is that the validity of the classical *lower-dimensional equations*, such as the two-dimensional von Kármán equations for nonlinearly elastic plates or the two-dimensional Koiter equations for linearly elastic shells, is no longer left unquestioned. A need has been felt for a better assessment of their relation to the corresponding three-dimensional equations that they are supposed to "replace".

Thanks to the ever-increasing power of available computers, sophisticated mathematical models that were previously intractable by *approximate methods* are now amenable to numerical simulations. This is one more reason why these models should be established on firm grounds.

This treatise illustrates at length these recent trends, as shown by the *main topics covered*:

- A thorough *description*, with a pervading emphasis on the *nonlinear aspects*, of the two existing mathematical models of *three-dimensional elasticity*, either as a *boundary value problem* consisting of a system of *three quasilinear partial differential equations* of

[1] This "General preface" is an updated excerpt from the "Preface" to the first edition (1988) of Volume I.

the *second order* together with specific *boundary conditions*, or as a *minimization problem* for the associated *energy* over an *ad hoc* set of *admissible deformations* (Vol. I, Part A);

- A *mathematical analysis* of these models, comprising in particular *complete proofs of all the available existence results*, relying either on the *implicit function theorem*, or on the *direct methods of the calculus of variations* (Vol. I, Part B);

- A mathematical justification of the well-known two-dimensional *linear Kirchhoff-Love theory of plates*, by means of *convergence theorems* as the thickness of the plate approaches zero (Vol. II, Part A);

- Similar justifications of mathematical models of *junctions in linearly elastic multi-structures* and of *linearly elastic shallow shells* (Vol. II, Part A);

- A systematic *derivation of two-dimensional plate models from nonlinear three-dimensional elasticity* by means of the *method of formal asymptotic expansions*, which includes a justification of well-known plate models, such as the *nonlinear Kirchhoff-Love theory* and the *von Kármán equations* (Vol. II, Part B);

- A derivation of the *large deformation, frame-indifferent, nonlinear planar membrane and flexural theories* by means of the method of formal asymptotic expansions and a justification of *nonlinear planar membrane equations* by means of a *convergence theorem* (Vol. II, Part B);

- A *mathematical analysis of the two-dimensional, linear and nonlinear, plate equations*, which includes in particular a review of the *existence* and *regularity theorems* in the nonlinear case and an introduction to *bifurcation theory* (Vol. II, Parts A and B);

- A mathematical justification by means of *convergence theorems* of the *two-dimensional membrane, flexural,* and *Koiter equations of a linearly elastic shell* (Vol. III, Part A);

- A systematic derivation of the *two-dimensional membrane* and *flexural equations of a nonlinearly elastic shell* by means of the *method of formal asymptotic expansions* and a justification of *nonlinear membrane shell equations* by means of a *convergence theorem* (Vol. III, Part B).

- A *mathematical analysis of the two-dimensional, linear and nonlinear, shell equations,* with a particular emphasis on the *existence* theory (Vol. III, Parts A and B).

Although the emphasis is definitely on the mathematical side, every effort has been made to keep the prerequisites, whether from mathematics or continuum mechanics, to a minimum, notably by making this treatise as largely *self-contained* as possible. Its reading only presupposes some familiarity with basic topics from analysis and functional analysis.

Naturally, frequent references are made to Vol. I in Vol. II, and to Vols. I and II in Vol. III. However, I have also tried to render *each volume as self-contained as possible*. In particular, all relevant notions from three-dimensional elasticity are (at least briefly) recalled wherever they are needed in Vols. II and III.

References are also made to Vol. I regarding various *mathematical notions* (properties of domains in $\mathbb{R}^n$, differential calculus in normed vector spaces, Sobolev spaces, weak lower semi-continuity, etc.). This is a mere convenience, reflecting that I also regard the three volumes as forming a coherent whole. I am otherwise well aware that Vol. I is neither a text on analysis nor one on functional analysis. Any reader interested in a deeper understanding of such notions should consult the more standard texts referred to in Vol. I.

Each volume is divided into consecutively numbered *chapters*. Chapter m contains an introduction, several sections numbered Sect. $m.1$, Sect. $m.2$, etc., and is concluded by a set of *exercises*. Within Sect. $m.n$, *theorems* are consecutively numbered, as Thm. $m.n$-1, Thm. $m.n$-2, etc., and *figures* are likewise consecutively numbered, as Fig. $m.n$-1, Fig. $m.n$-2, etc. *Remarks* and *formulas* are not numbered. The end of the proof of a theorem, or the end of a remark, is indicated by the symbol ■ in the right margin. In Chapter m, exercises are numbered as Ex. $m.1$, Ex. $m.2$, etc.

All the important results are stated in the form of *theorems* (there are no lemmas, propositions, or corollaries), which therefore represent the core of the text. At the other extreme, the *remarks* are intended to point out some interpretations, extensions, counter-examples, relations with other results, that in principle can be skipped during a first reading; yet, they could be helpful for a better understanding of the material. When a term is defined, it is set in boldface if it is deemed important, or in italics otherwise. Terms that are only given a loose or intuitive meaning are put between quotation marks.

Special attention has been given to the *notation*, which so often has a distractive and depressing effect in a first encounter with elasticity. In particular, each volume begins with special sections, which the reader is urged to consult first, about the notations and the rules that have guided their choice. The same sections also review the main *definitions* and *formulas* that will be used throughout the text.

Complete proofs are generally given. In particular, whenever a mathematical result is of particular significance in elasticity, its proof has been included. More standard mathematical prerequisites are presented (usually without proofs) in special *starred sections*, scattered according to the local needs. The proofs of some *advanced, or more specialized, topics*, are sometimes only sketched, in order to keep the length of each volume within reasonable limits; in this case, *ad hoc* references are always provided. These topics are assembled in special sections marked with the symbol $^\flat$, usually at the end of a chapter.

Exercises of varying difficulty are included at the end of each chapter. Some are straightforward applications of, or complements to, the text; others, which are more challenging, are usually provided with hints or references.

This treatise would have never seen the light had I not had the good fortune of having met, and worked with, many exceptional students and colleagues, who helped me over the past three decades decipher the arcane subtleties of mathematical elasticity; their names are listed in the preface to each volume. To all of them, my heartfelt thanks!

I am also particularly indebted to Arjen Sevenster, whose constant interest and understanding were an invaluable help in this seemingly endless enterprise!

Last but not least, this treatise is dedicated to Jacques-Louis Lions, as an expression of my deep appreciation and gratitude.

August, 1986 and October, 1999 Philippe G. Ciarlet

PREFACE TO VOLUME I[1]

A fascinating aspect of three-dimensional elasticity is that, in the course of its study, one *naturally* feels the need for studying *basic mathematical techniques* of *matrix theory, analysis*, and *functional analysis*; how could one find a better motivation? For instance:

- Both common and uncommon results from *matrix theory* are often needed, such as the polar factorization theorem (Thm. 3.2-2), or the celebrated Rivlin-Ericksen representation theorem (Thm. 3.6-1). In the same spirit, who would think that the inequality $|\operatorname{tr} \mathbf{AB}| \leq \sum_i v_i(\mathbf{A})v_i(\mathbf{B})$, where $v_i(\mathbf{A})$ and $v_i(\mathbf{B})$ denote the singular values, arranged in increasing order, of the matrices $\mathbf{A}$ and $\mathbf{B}$, arises naturally in the analysis of a wide class of actual stored energy functions? Incidentally, this seemingly innocuous inequality is not easy to prove (Thm. 3.2-4)!

- The understanding of the "geometry of deformations" relies on a perhaps elementary, but "applicable", knowledge of *differential geometry.* For instance, my experience is that, among those of my students who had been previously exposed to modern differential geometry, very few could effectively produce the formula $da^{\varphi} = |\mathbf{Cof}\,\boldsymbol{\nabla}\boldsymbol{\varphi}\,\boldsymbol{n}|da$ relating reference and deformed area elements (Thm. 1.7-1).

- The study of geometrical properties (orientation-preserving character, injectivity) of mappings in $\mathbb{R}^3$ naturally leads to using such basic tools as the *invariance of domain theorem* (Thms. 1.2-5 and 1.2-6) or the *topological degree* (Sect. 5.4); yet these are unfortunately all too often left out from standard analysis courses.

- *Differential calculus in Banach* spaces is an indispensable tool which is used throughout this volume, and the unaccustomed reader should quickly become convinced of the many merits of the *Fréchet derivative* and of the *implicit function theorem*, which are the keystones to the existence theory developed in Chap. 6.

- The fundamental *Cauchy-Lipschitz existence theorem for ordinary differential equations in Banach spaces*, as well as the conver-

[1] This "Preface to Volume I" is an updated excerpt from the "Preface" to the first edition (1988) of Volume I.

gence of its *approximation by Euler's method*, are needed in the analysis of incremental methods, often used in the numerical approximation of the equations for nonlinearly elastic structures (Chap. 6).

- Basic topics from *functional analysis* and the *calculus of variations*, such as *Sobolev spaces* (which in elasticity are simply the "spaces of finite energy"), *weak convergence, existence of minimizers for weakly lower semi-continuous functionals*, pervade the treatment of existence results in three-dimensional elasticity (Chaps. 6 and 7).

- Key results about *elliptic linear systems of partial differential equations*, notably sufficient conditions for the $W^{2,p}(\Omega)$-*regularity* of their solutions (Thm. 6.3-6), are needed preliminaries for the existence theory of Chap. 6.

- As a result of John Ball's seminal work in three-dimensional elasticity, *convexity* and the subtler *polyconvexity* play a particularly important rôle throughout this volume. In particular, we shall naturally be led to finding nontrivial examples of *convex hulls*, such as that of the set of all square matrices whose determinant is > 0 (Thm. 4.7-4), and of *convex functions of matrices.* For instance, functions such as $\boldsymbol{F} \to \sum_i \{\lambda_i(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F})\}^{\alpha/2}$ with $\alpha \geq 1$ naturally arise in the study of *Ogden's materials* in Chap. 4; while proving that such functions are convex is elementary for $\alpha = 2$, it becomes surprisingly difficult for the other values of $\alpha \geq 1$ (Sect. 4.9). Such functions are examples of John Ball's *polyconvex stored energy functions*, a concept of major importance in elasticity (Chaps. 4 and 7).

- In Chap. 7, we shall come across the notion of *compensated compactness.* This technique, discovered and studied by François Murat and Luc Tartar, is now recognized as a powerful tool for studying nonlinear partial differential equations.

Another fascinating aspect of three-dimensional elasticity is that it gives rise to a number of *open problems*, for instance:

- The extension of the "local" analysis of Chap. 6 (existence theory, continuation of the solution as the forces increase, analysis of incremental methods) to genuine mixed displacement-traction problems;

- "Filling the gap" between the existence results based on the implicit function theorem (Chap. 6) and the existence results based on the minimization of the energy (Chap. 7);

- An analysis of the nonuniqueness of solutions (cf. the examples given in Sect. 5.8);

- A mathematical analysis of contact with friction (contact, or self-contact, without friction is studied in Chaps. 5 and 7);

- Finding reasonable conditions under which the minimizers of the energy (Chap. 7) are solutions of the associated Euler-Lagrange equations;

- While substantial progress has been made in the *study of statics* (which is all that I consider here), the analysis of time-dependent elasticity is still at an early stage. Deep results have been recently obtained for one space variable, but formidable difficulties stand in the way of further progress in this area.

This volume will have fulfilled its purposes if the above messages have been conveyed to its readers, that is,

- if it has convinced its more application-minded readers, such as continuum mechanicists, engineers, "applied" mathematicians, that mathematical analysis is an indispensable tool for a genuine understanding of three-dimensional elasticity, whether it be for its modeling or for its analysis, essentially because more and more emphasis is put on the nonlinearities (e.g., injectivity of deformations, polyconvexity, nonuniqueness of solutions, etc.), whose consideration requires, even at the onset, some degree of mathematical sophistication;

- if it has convinced its more mathematically oriented readers that three-dimensional elasticity, far from being a dusty classical field, is on the contrary a prodigious source of challenging open problems.

Although more than 570 items are listed in the bibliography, there has been no attempt to compile an exhaustive list of references. The interested readers should look at the extensive bibliography covering the years 1678-1965 in the treatise of Truesdell & Noll [1965], at the additional references found in the books by Marsden & Hughes [1983], Hanyga [1985], Oden [1986], and especially Antman [1995], and in the papers of Antman [1983] and Truesdell [1983], which give short and illuminating historical perspectives on the interplay between elasticity and analysis.

The readers of this volume are strongly advised to complement the material given here by consulting a few other books, and in this respect, I particularly recommend the following *general references on three-dimensional elasticity* (general references on lower-dimensional theories of plates, shells and rods are given in Vols. II and III):

- In-depth perspectives in continuum mechanics in general, and in elasticity in particular: The treatises of Truesdell & Toupin [1960] and Truesdell & Noll [1965], and the books by Germain [1972], Gurtin [1981b], Eringen [1962], and Truesdell [1991].

- Classical and modern expositions of elasticity: Love [1927], Murnaghan [1951], Timoshenko [1951], Novozhilov [1953], Sokolnikoff [1956], Novozhilov [1961], Landau & Lifchitz [1967], Green & Zerna [1968], Stoker [1968], Green & Adkins [1970], Knops & Payne [1971], Duvaut & Lions [1972], Fichera [1972a, 1972b], Gurtin [1972], Wang & Truesdell [1973], Villaggio [1977], Gurtin [1981a], Nečas & Hlaváček [1981], and Ogden [1984].

- Mathematically oriented treatments in nonlinear elasticity: Marsden & Hughes [1983], Hanyga [1985], Oden [1986], and the landmark book of Antman [1995].

- The comprehensive survey of numerical methods in nonlinear three-dimensional elasticity of Le Tallec [1994].

In my description of continuum mechanics and elasticity, I have only singled out *two axioms*: The stress principle of Euler and Cauchy (Sect. 2.2) and the axiom of material frame-indifference (Sect. 3.3), thus considering that all the other notions are *a priori* given. The reader interested in a more axiomatic treatment of the basic concepts, such as frame of reference, body, reference configuration, mass, forces, material frame-indifference, isotropy, should consult the treatise of Truesdell & Noll [1965], the books of Wang & Truesdell [1973] and Truesdell [1991], and the fundamental contributions of Noll [1959, 1966, 1972, 1973, 1978].

At the risk of raising the eyebrows of some of my readers, and at the expense of various *abus de langage*, I have also ignored in this volume the difference between second-order tensors and matrices. The readers disturbed by this approach should look at the books of Abraham, Marsden & Ratiu [1983] and, especially, of Marsden & Hughes [1983], where they will find all the tensorial and differential geometric aspects of elasticity explained in depth and put in their proper perspective. Likewise, Vol. III should be also helpful in this respect.

This volume is an outgrowth of lectures on elasticity that I have given over the past 15 years at the Tata Institute of Fundamental Research, the University of Stuttgart, the Ecole Normale Supérieure,

and the Université Pierre et Marie Curie.

I am particularly indebted to the many students and colleagues I worked with on that subject during the same period; in particular: Michel Bernadou, Dominique Blanchard, Jean-Louis Davet, Philippe Destuynder, Giuseppe Geymonat, Hu Jian-wei, Srinivasan Kesavan, Klaus Kirchgässner, Florian Laurent, Hervé Le Dret, Jindřich Nečas, Robert Nzengwa, Jean-Claude Paumier, Peregrina Quintela-Estevez, Patrick Rabier, and Annie Raoult.

Special thanks are also due to Stuart Antman, Irene Fonseca, Morton Gurtin, Patrick Le Tallec, Bernadette Miara, François Murat, Tinsley Oden, and Gérard Tronel, who were kind enough to read early drafts of this volume and to suggest significant improvements.

For their especially expert and diligent assistance as regards the material realization of this volume, I very sincerely thank Hélène Bugler, Monique Damperat, and Liliane Ruprecht.

August, 1986 and October, 1999 Philippe G. Ciarlet

PREFACE TO VOLUME II [1]

Lower-dimensional plate, shell, and *rod, theories* that rely on *a priori* assumptions of a mechanical or geometrical nature have been proposed by A.-L. Cauchy, Sophie Germain, G. Kirchhoff, T. von Kármán, A.E.H. Love, E. Reissner, Jakob Bernoulli, C.-L.-M.-H. Navier, L. Euler, S.-D. Poisson, E. and F. Cosserat, L.H. Donnell, W. Flügge, S.P. Timoshenko, V.V. Novozhilov, I.N. Vekua, A.E. Green, W.T. Koiter, J.G. Simmonds, P.M. Naghdi, and others.

There are two reasons why these lower-dimensional theories are so often preferred to the three-dimensional theory that they are supposed to "replace" when the thickness, or the diameter of the cross-section, is "small enough".

One reason is their *simpler mathematical structure*, which in turn generates a richer variety of results. For instance, the existence, regularity, or bifurcation, theories, and more generally the "global analysis", are by now on firm mathematical grounds for nonlinearly elastic rods (see Antman [1995] for a scholarly and comprehensive exposition) or for nonlinearly elastic von Kármán plates (see Ciarlet & Rabier [1980]). By contrast, these theories of global analysis are still partly in their infancies for nonlinear three-dimensional elasticity (see Marsden & Hughes [1983] and Vol. I for comprehensive surveys): After the fundamental ideas set forth by Ball [1977], who was able to establish the existence of a minimizer of the energy for a wide class of realistic nonlinearly elastic materials, there indeed remain manifold challenging open problems; for instance, there is no known set of sufficient conditions guaranteeing that such a minimizer satisfies the equilibrium equations even in the weak sense of the principle of virtual work (another existence theory, based on the implicit function theorem, does not share this drawback, but it is restricted to problems with smooth data and to special boundary conditions, unrealistic in practice; see Vol. I and the comprehensive treatment of Valent [1988]).

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[1] A substantial portion of this preface is an excerpt from the "Introduction" in Ciarlet & Lods [1996b].