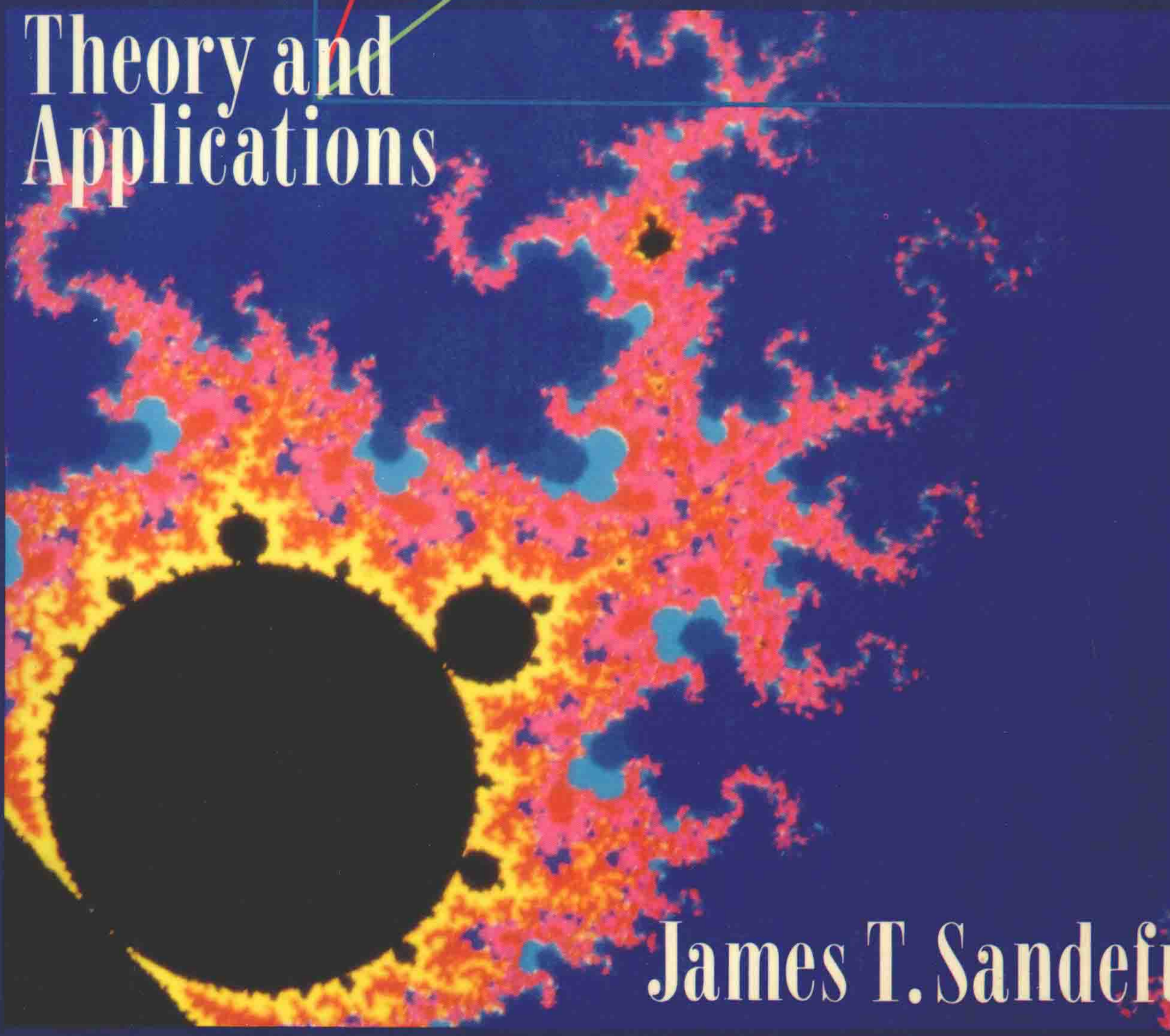
DISCRETE DYNAMICAL SYSTEMS

Theory and Applications



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To Helen

Preface

The study of dynamics is the study of how things change over time. Discrete dynamics is the study of quantities that change at discrete points in time, such as the size of a population from one year to the next, or the change in the genetic make-up of a population from one generation to the next. In general, we concurrently develop a model of some situation and the mathematical theory necessary to analyze that model. As we develop our mathematical theory, we will be able add more components to our model.

The means for studying change is to find a relationship between what is happening now and what will happen in the 'near' future; that is, cause and effect. By analyzing this relationship, we can often predict what will happen in the distant future. The distant future is sometimes a given point in time, but more often is a limit as time goes to infinity. In doing our analysis, we will be using many algebraic topics such as, factoring, exponentials and logarithms, solving systems of equations, manipulating imaginary numbers, and matrix algebra. We will also use topics from calculus, such as derivatives and graphing techniques. The mathematical theory generally builds on results developed earlier in this text.

After reading this text, you should be able apply discrete dynamics to any field in which things change, which is most fields. The goal, then, is to not only learn mathematics, but to get develop a differently way of thinking about the world.

My own interest in this material is somewhat backwards. Several years ago I became interested in a topic of current mathematical research, chaos. One result of the theory of chaos is that there are certain situations that change over time in an apparently random manner, and no amount of analysis will enable us to make accurate predictions for more than a short period of time. While this topic is extremely complicated, the ideas behind it can be presented without a lot of mathematical background. As I studied these chaotic models, I came to learn more and more about situations in which we **can** make accurate long term predictions. In fact, discrete dynamics has a long and useful history in many fields of study, but has been largely ignored by mathematicians until the recent interest in chaos.

My idea was to write a text that shows the many cases in which mathematics succeeds in its ability to make predictions. By understanding these cases, we will have a greater appreciation of situations in which math sometimes 'fails', that is, chaos. Thus, while chaos is discussed in this text, it is introduced as a later step in the mathematical understanding of the world.

Acknowledgements

As with any multi-year project, there are many people to thank. I would like to thank all my colleagues at Georgetown University and, in particular, Professor Ronald Rosier. It was through a summer program he directed that I got the idea for this text. His continued support and encouragement over the years has been invaluable. I would also like to thank Ms. Anita Dickerson for typing much of this manuscript from nearly illegible notes, and Professor A. Paul Stokes for his help in producing many of the figures in this text. I am grateful to those who read my manuscript and gave me helpful suggestions. These include Professors Ray Bobo, Andrew Vogt, Robert Devaney, and David Hart.

While many students endured my lectures as I developed the ideas in this text, I would particularly like to thank three: Mr. Lawrence Letellier and Mr. Tony Pribadi who gave me the idea on the gambler's ruin, and Ms. Pamela McGuire who gave me some ideas in genetics.

Finally, I would like to thank my wife, Helen Moriarty, without whose support this project would never have been finished.

Notes to the teacher

While it is not necessary to have a calculator or computer to understand this material, it would be helpful. With the aid of a calculator or computer, students can study many complex and interesting applications right from the start. Once a simple mathematical relationship is found, students can easily run their own math experiments and make their own hypotheses about what will happen. The verification of these hypotheses will have to wait until the appropriate theoretical model is developed.

Some of the earlier material in the book can be carried out using a computer spreadsheet. Experiments can be run on spreadsheets, such as guessing a monthly payment on a loan that over a given period of time makes the end amount owed equal to zero. Spreadsheets can also be used to make graphs similar to some graphs in this text.

Chapters 1 and 2 are essential for the rest of the text (excluding Section 1.6). Chapters 3, 5, and 6 could be studied in any order, but Chapter 4 depends on Chapter 3. Chapter 7 depends on Chapters 3, 4, and 6.

Applications are for the most part independent of one another. Thus, particular applications can be omitted.

The easiest method for compiling tests is to pick what you want as an answer, then work backwards to get a question with that answer. I must admit that it is difficult to develop good modeling questions.

Contents

Introduction to dynamical systems

1

1.1 Introduction to discrete dynamical modeling

Dynamical modeling is the art of modeling phenomena that change over time. The normal procedure we use for creating a model is as follows: first we identify a real world situation that we wish to study and make assumptions about this situation. Second, we translate our assumptions into a mathematical relationship. Third, we use our knowledge of mathematics to analyze or 'solve' this relationship. Fourth, we translate our solution back into the real world situation to learn more about our original model.

There are two warnings. First, the mathematical relationship is not the solution. For example, suppose we wish to model a square that has an area of 4 square meters. One mathematical translation is $x^2 = 4$, where x is the length of one side. Notice that this is not a solution. Analysis gives the possible solutions, $x = 2$ and $x = -2$. The second warning is to make sure that the solution makes sense in the situation being considered. In the example of our square, $x = -2$ does not make sense. Translating the solution $x = 2$ back to the real world, we learn something about our square, that is, its sides are of length 2 meters.

Often, none of the mathematical solutions makes sense, so the original assumptions must be reconsidered (or the mathematical calculations need to be rechecked).

In this text, we will consider situations in which the state of the system at one point in time depends on the state of the system at previous points in time.

Example 1.1 Suppose we start a savings account of 1000 dollars on January 1, 1983, and that the bank pays 10 per cent interest on its accounts, compounded annually. Then on January 1, 1984, we will have 1100 dollars in our account (our original 1000 dollars plus 10 per cent of our 1000 dollars as interest). Now the 'interesting' thing about this model is that on January 1, 1985, we do not have 1200 dollars, but to our satisfaction we have 1210 dollars in our account. This is the 1100 dollars we already have plus (0.10)(1100) – our 10 per cent interest rate times our balance. Note that (a) knowing what is true today (we have 1000 dollars in the bank) and (b) having knowledge about the world that relates today to some day in the future (1 year from today we will have 10 per cent more money), we can predict how much money we will have at any time in the future (barring some unknown factor such as taxes, changes in interest rates or a bank failure).

Let's now translate this situation into the language of mathematics. This is not difficult as long as you remember the definitions and you always write in complete sentences. We will let January 1, 1983, be time 0; January 1, 1984, will be time 1; January 1, 1985, will be time 2; etc.

So January 1, 2001, will be time $2001 - 1983 = 18$. All other days are irrelevant, since the money in our account stays fixed until January 1 of the next year. We now let $A(0)$ be the amount (in dollars) that we have in our account at time 0, that is, $A(0) = 1000$. Likewise $A(1) = 1100$ and $A(2) = 1210$. Our goal will be to **predict** how much we will have in our account at time n where n is some future year. For example, if $n = 18$ (year 2001), what is $A(18)$?

We are now able to write our savings account problem as

$$A(1) = A(0) + (0.1)A(0)$$

which is read as: the amount at time 1 is (or equals) the amount at time 0 plus interest on the amount at time 0 (10 per cent of $A(0)$). This can be simplified to $A(1) = (1.1)A(0)$. Likewise

$$A(2) = (1.1)A(1), \quad A(3) = (1.1)A(2), \quad \ldots, \quad A(18) = (1.1)A(17),$$

and so forth.

We need a shorthand expression for the above equations. Verbally, the equations say that **the amount in our account next year is the amount in our account this year plus interest on the amount this year.** Let this year be year n. Then next year is year $n + 1$. The amount in the bank each of these years is denoted $A(n)$ and $A(n + 1)$, respectively. Thus the statement above is read mathematically as

$$A(n+1) = A(n) + (0.1)A(n), \quad \text{for } n = 0, 1, 2, 3, \ldots . \tag{1}$$

Note that equation (1) when read aloud in a complete sentence actually reads the same as the boldface statement above. Note that this does not solve our problem, but restates it mathematically. To solve our problem, we need some method of finding the amount in our account at any time in the future.

One method for finding a 'solution' would be direct computation. Suppose we want to find $A(18)$. Since $A(0) = 1000$, by substitution we get that

$$A(1) = (1.1)A(0) = 1100.$$

Repeating this process, we get that

$$A(2) = 1.1A(1) = (1.1)1100 = 1210, \quad A(3) = (1.1)A(2) = 1331,$$

and so forth.

Before the days of computers, it would be tedious and time consuming to compute $A(18)$, but recursive tasks are what computers do best. Thus, after writing a simple program, we could compute $A(100)$ or $A(1000)$ quickly. Many of the results in this text were done on a computer. The programs are all easy to write in almost any computer language (and can also be written on a programmable/graphics calculator). Spreadsheets are also quite useful in performing recursive tasks. While you will not need access to a computer/calculator to follow this material, if you do use one you will have fun and learn a lot by computing answers to problems similar to the ones stated. For example, you could compute $A(1), A(2), \ldots$ when $A(0) = 100$ (or some other number of your choice).

In an effort to find a 'better' solution, that is, one that is easier to work with, we make the substitutions

$$A(2) = 1.1A(1) = 1.1(1.1A(0)) = (1.1)^2 A(0)$$

$$A(3) = 1.1A(2) = 1.1((1.1)^2 A(0)) = (1.1)^3 A(0), \ldots .$$

From this, it is not difficult to see that

$$A(k) = (1.1)^k A(0), \quad \text{for } k = 1, 2, \ldots . \tag{2}$$

Equation (2) is what we mean by a solution to our problem. We can use this equation and a calculator to compute easily the amount we have in our account at any future point in time.

The above example is indicative of the rest of this text, in the sense that we will use our knowledge of today to make predictions about tomorrow, then we will use our predictions about tomorrow to make predictions about the day after that, and so forth. Using this simple iterative idea, we will gain insight about the way the world operates. Particular examples will be from population growth, genetics, economics, and gambling, to name just a few.

Once we have analyzed a problem, the next step is to generalize. The reason for this is that generalizations are often as easy to study as the particular example while being far more widely applicable.

To generalize our savings problem, suppose the interest was not 10 per cent but some other per cent, say $100I$ per cent (where $I = 0.1$ in our previous discussion). The amount next year is the amount we have this year plus I times the amount we have this year. Written in the language of mathematics

$$A(n+1) = A(n) + IA(n) = (1+I)A(n), \qquad \text{for } n = 0, 1, 2, \ldots . \tag{3}$$

Note that when $I = 0.1$, equation (3) is the same as equation (1).

Mathematicians prefer writing equations in more general forms, like equation (3), not because we like to confuse people, but because we are lazy (in the sense that we do not want to do work that is unnecessary). It actually happens that it is as easy to handle equation (3) as it is to handle equation (1). But once we have analyzed equation (3), if the bank changes its interest rate we will know what happens to our money **with no additional work.**

Similar to Example 1.1, the solution to equation (3) is

$$A(k) = (1 + I)^k A(0), \qquad \text{for } k = 0, 1, 2, \ldots. \tag{4}$$

Equations that describe a relationship between one point in time and a previous point in time, such as equation (1) and equation (3), are called **discrete dynamical systems** or **difference equations**. Closed form, or explicit, expressions for the amount in any year, such as equation (2) and equation (4), are called **solutions** to the corresponding dynamical system. This will be stated more precisely in the next section.

1.1.1 Problems

1. Suppose a bank pays 5 per cent interest on its savings accounts, compounded annually.

   (a) Write down a dynamical system for the amount in the account in year $n + 1$ using the amount in the account in year n.

   (b) Given that the initial deposit is $A(0) = 200$, find the amount in the account after 1, 2, 3, and 4 years.

   (c) Give the solution to the dynamical system.

2. Suppose a broker charges a 2 per cent service charge on the money in your savings account each year.

   (a) Assuming your broker makes bad investments and that you do not earn any interest on your account, give a dynamical system for the amount in your account one year using the amount in the account the previous year.

   (b) Given that the initial deposit is 500 dollars, find the amount in your account after 1, 2, 3, and 4 years.

   (c) Give the solution to the dynamical system.

3. Suppose a bank pays 8 per cent interest each year on its checking accounts, but it also deducts 40 dollars per year as a service charge (after first adding on the interest).

   (a) If your initial deposit is 1000 dollars, what do you have in your account after 1 year, 2 years, and 3 years?

(b) Write a dynamical system to model this process.

4. Suppose that you borrow 2000 dollars from a friend. You agree to add 1 per cent interest each month to the amount of the loan that is still outstanding and also to pay your friend 150 dollars each month. Assume that the interest is first added on to what you owe and then your 150 dollar payment is subtracted.

   (a) Write a dynamical system to describe the amount $A(n)$ that you owe your friend after n months.

   (b) Using a calculator and the dynamical system, find how many months it will take you to pay off your loan and what the final payment will be.

5. Suppose your bank pays 8 per cent interest, compounded quarterly, and you initially deposit 100 dollars. Let $A(n)$ be the amount in your account after n quarters.

   (a) Develop a dynamical system to describe the amount in your account in one quarter in terms of the amount in the previous quarter.

   (b) Compute the amount in your account at the end of 1 year. Notice that you made more than 8 per cent interest for the year. (Remember that $A(1)$ is the amount after one quarter, not 1 year.)

   (c) Develop a closed form expression for the amount in your account after n quarters.

   (d) Develop a closed form expression for the amount in your account after t years.

6. Suppose your bank pays $100I$ per cent interest, compounded m times per year. Let $A(n)$ be the amount in your account after n compounding periods. For example, $A(m)$ represents the amount after 1 year.

   (a) Develop a dynamical system to describe the amount in your account in one compounding period in terms of the amount in the previous compounding period.

   (b) Develop a closed form expression for the amount in your account after n compounding periods in terms of I, $A(0)$, and m.

(c) Develop a closed form expression for the amount in your account after t years in terms of I, $A(0)$, and m.

1.2 Terminology

The dynamical systems we will consider come in many different forms but, as we will see, seemingly different types of equations can be handled similarly. Therefore we will divide these equations into large classes and study each class separately.

Informally, a discrete dynamical system is a sequence of numbers that are defined recursively, that is, there is a rule relating each number in the sequence to previous numbers in the sequence. One example is the sequence 0, 1, 2, ... Denoting each of these numbers by $A(k) = k$ for $k = 0, 1, 2, \ldots$, we note that the rule relating the numbers is $A(n+1) = A(n) + 1$. For the sequence 2, 4, 8, 16, ..., the rule is $A(n+1) = 2A(n)$, that is, each number is twice the previous number.

It is usually easier to give the rule and the first number, and then compute the sequence. Consider the rule

$$A(n+1) = 2A(n)(1 - A(n)),$$

with the first number being $A(0) = 0.1$. We then get the sequence

$$A(1) = 2A(0)(1 - A(0)) = 0.18, \quad A(2) = 0.2952, \quad A(3) = 0.416, \ldots .$$

Definition 1.2 *Suppose we have a function $y = f(x)$. A* **first order discrete dynamical system** *is a sequence of numbers $A(n)$ for $n = 0, \ldots$ such that each number after the first one is related to the previous number by the relation*

$$A(n+1) = f(A(n)).$$

The sequence of numbers given by the relationship

$$A(n+1) - A(n) = g(A(n))$$

is called a **first order difference equation**. *Note that by letting $f(x) = g(x) + x$, these two concepts are seen to be equivalent.*

From now on, we will often omit the term 'discrete' and just call such sequences **dynamical systems**. We will also equate a dynamical system with the rule that defines it. Three examples of dynamical systems are the relationships

$$A(n+1) = 3A(n), \quad A(n+1) = 2A(n) + 5, \quad A(n+1) = \frac{A(n)}{1 + A(n)}.$$