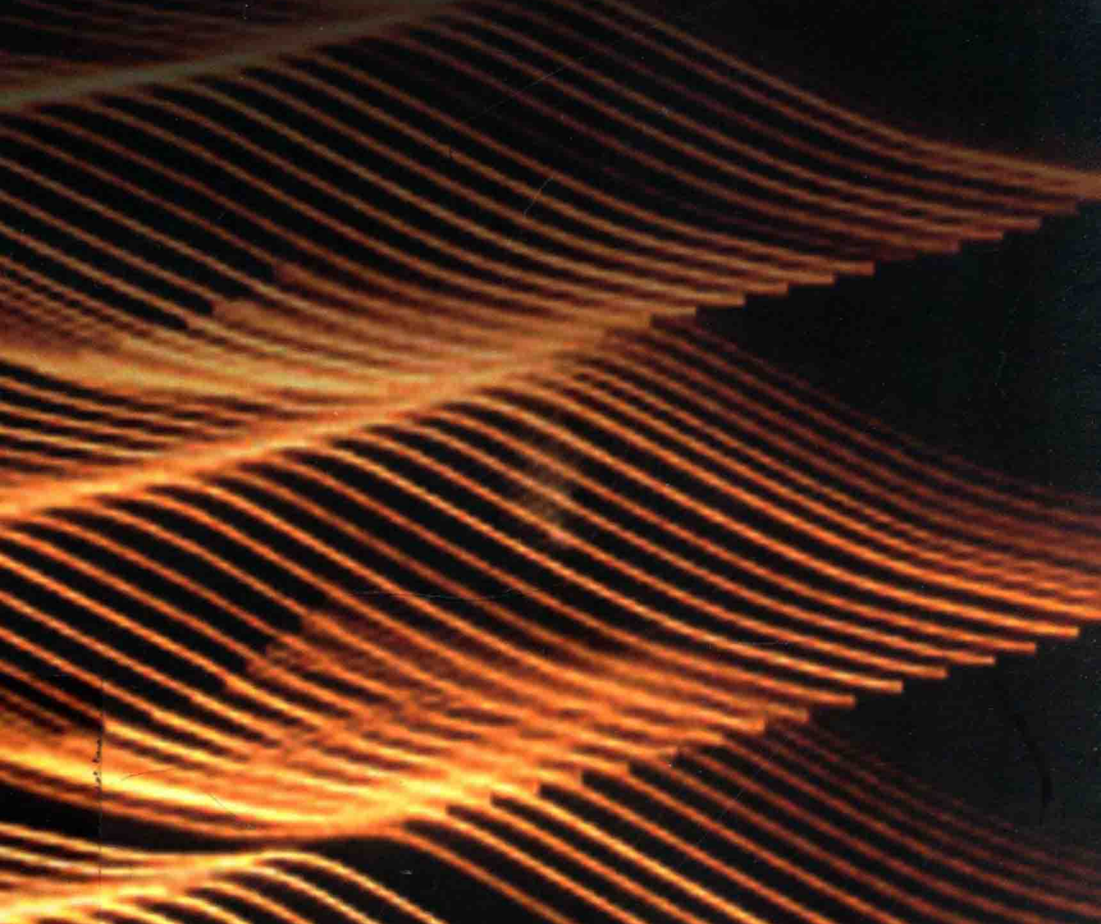
FUZZY MATRIX

THEORY AND APPLICATIONS

A R MEENAKSHI



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Dr A R Meenakshi
AICTE-Emeritus Professor of Mathematics
Dean of Mathematics and Computer Applications
Karpagam University
Coimbatore

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PREFACE

This is a first attempt to survey fuzzy matrices, the theory related to them and their applications to various sciences. Most of the results have appeared in various journals and in the midst of other related topics on fuzzy sets, fuzzy logic and information sciences. A development on the theory of fuzzy matrices is made analogous to that of Boolean matrices. Since binary fuzzy relations are represented as fuzzy matrices, applications of the theory of fuzzy matrices are of fundamental importance in the formulation and analysis of many classes of discrete structural models which arise in physical, biological, medical, social and engineering sciences. A fuzzy matrix is a matrix over a fuzzy algebra and the algebraic operations on fuzzy matrices are max–min operations, which are different from that of the standard operations on real and complex matrices. Therefore fuzzy matrices require a completely separate treatment from that of matrices over the real field or complex field, due to which fuzzy matrices do not satisfy many of the fundamental properties of real or complex matrices. For instance, a fuzzy matrix need not always possess a generalized g-inverse, in sharp contrast to the real or complex cases, where every matrix has a g-inverse. Also, for a fuzzy matrix, row and column rank need not be equal, which is again a fundamental property for real or complex matrices. The application of g-inverses are rapidly increasing and a regular matrix for which these fundamental properties hold good has gained much importance in fuzzy sets theory. These fundamental properties hold. For a fuzzy matrix A, $A^{k+d} = A^k$ for some positive integers k, d; therefore all fuzzy matrices have an index and a period. On the other hand most matrices over the non-negative real numbers will not have an index and a period. Due to these interesting deviations from the general complex field, a few of the usual definitions for the real (complex) field need modifications to be meaningful for fuzzy matrices.

The material covered in the book mostly relates to the research work done during the last 10 years and a number of unpublished results recently obtained by the author. The manuscript is an outgrowth of lectures given at Annamalai University, India, and for refresher/summer courses organized by various universities in Tamil Nadu. This book is intended to provide an understanding of the mathematical models involving fuzzy matrices to students and researchers in a broad range of disciplines. It is written specifically as a text for one semester course for graduate students in mathematics, engineering, computer application disciplines and biological Sciences. The book would be useful as supplementary material in a variety of courses such as fuzzy mathematics, fuzzy logic, control theory and information Sciences. No previous knowledge of fuzzy set theory or information theory is required for an understanding of the material in this book. However we assume that the reader is familiar with the basic notions of classical set theory and matrix theory.

In chapter I, the fundamental concepts such as fuzzy vectors, fuzzy vector spaces and standard basis are presented and the various ranks associated with a fuzzy matrix are defined and studied. Green's relations for fuzzy matrices are discussed.

In chapter II special types of fuzzy relations are introduced and consistency of fuzzy relational equations is discussed. The determinant theory of fuzzy matrices is explained. In Chapter III, various generalized inverses of a fuzzy matrix are defined, conditions for their existence are derived and characterizations of the set of generalized inverses are obtained.

Chapter IV deals with partial orderings on fuzzy matrices. Chapter V deals with block fuzzy matrices and discuss the regularity and decomposition of a block fuzzy matrix in terms of the Schur complements are discussed. In chapter VI some applications of fuzzy matrices in document retrieval system, medical diagnosis, database management system, decision making theory and dynamical systems are outlined.

Chapter VII deals with the study of fuzzy linear transformations on finite dimensional fuzzy vector spaces. One-to-one correspondence between fuzzy linear transformation and fuzzy matrices is established. The concepts of fuzzy matrices and its various applications are developed iteratively and illustrated with suitable examples wherever necessary.

At the end of each chapter brief notes and exercises are provided for the benefit of the students.

A R Meenakshi

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Any suggestions for the improvement of this book are most welcome.

AR Meenakshi

NOTATIONS

Throughout we are concerned with the fuzzy algebra $\mathcal{F}$ with support [0, 1]. $\alpha \in \mathcal{F}$ means that $a \in [0,1]$.

$\mathcal{F}_{mn}$	set of all $m \times n$ matrices over $\mathcal{F}$.
$\mathcal{F}_n$	set of all $n \times n$ matrices over $\mathcal{F}$.
$L(V, V')$	set of all linear transformations from V into V'.
$L(V)$	set of all linear transformations from V to itself.

For $A \in \mathcal{F}_{mn}$,

A^T	the transpose of A.
A^-	the g-inverse of A.
$A^\#$	the group inverse of A.
A_d	the Drazin inverse of A.
A^+	the Moore–Penrose inverse of A.
A_{i*}	the i^{th} row of A.
A_{*j}	the j^{th} column of A.
$\rho_r(A)$	the row rank of A.
$\rho_c(A)$	the column rank of A
$\mathbb{R}(A)$	the row space of A.
$\mathbb{C}(A)$	the column space of A.
$A\{1\}$	the set of all g-inverses of A.
$A^-_{\text{com}}\{1\}$	the set of all g-inverses of A which commute with A.
$\mathcal{F}^-_{mn}$	$\{A \in \mathcal{F}_{mn} / A$ has a g-inverse$\}$.
$\mathcal{F}^\#_n$	$\{A \in \mathcal{F}_n / A$ has group inverse$\}$.
$\mathcal{F}^+_{mn}$	$\{A \in \mathcal{F}_{mn} / A$ has Moore–Penrose inverse$\}$.
N	the set $\{1, 2, \cdots, n\}$
S_n	the symmetric group on $\{1, 2, \cdots, n\}$.

CONTENTS

1

FUNDAMENTAL CONCEPTS

This chapter is devoted to introduce the fundamental concepts of fuzzy vectors and fuzzy matrices. In order to develop the theory of fuzzy matrices, we begin with the concept of fuzzy algebra. A fuzzy algebra is a mathematical system $(F,+,\cdot)$ with two binary operations +, '$\cdot$' defined on a set F satisfying the following properties:

(P1)	Idempotence	$a + a = a$
		$a \cdot a = a$
(P2)	. Commutativity	$a + b = b + a$
		$a \cdot b = b \cdot a$
(P3)	Associativity	$a + (b + c) = (a + b) + c$
		$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
(P4)	Absorption	$a + (a \cdot b) = a$
		$a \cdot (a + b) = a$
(P5)	Distributivity	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
		$a + (b \cdot c) = (a + b) \cdot (a + c)$
(P6)	Universal bounds	$a + 0 = a$; $a + 1 = 1$
		$a \cdot 0 = 0$; $a \cdot 1 = a$

A Boolean algebra is a fuzzy algebra with complementarity, that is, for each element a, there exists an element $\overline{a}$, called the complement of a such that $a + \overline{a} = 1; a \cdot \overline{a} = 0$. Thus we have introduced the concept of fuzzy algebra as a generalization of a Boolean algebra (p. 24, [36]) (p. 1, [32]).

The following are examples of fuzzy algebra:

1. The unit closed interval $[0,1]$ of reals with the operations $(+, \cdot)$ defined as $a + b = \sup\{a, b\}$ and $a \cdot b = \inf\{a, b\}$ for all $a,b \in [0,1]$. This is called the max–min fuzzy algebra.
2. The unit closed interval $[0,1]$ of reals with the operations $(+, \cdot)$ defined as $a + b = \inf\{a, b\}$ and $a \cdot b = \sup\{a, b\}$ for all $a,b \in [0,1]$. This is called the min–max fuzzy algebra.
3. A distributive bounded lattice $(L, \wedge, V)$
4. A Brouwerian lattice $(L, \wedge, V)$ with universal bounds.
5. Two elements Boolean algebra $\{0, 1\}$ is a fuzzy algebra.

These fuzzy algebras are widely used in fuzzy set theory.

The characteristic function of a crisp set assigns a value of either 1(or) 0 to each individual in the universal set; thereby discriminating between members and non-members of the crisp set under consideration. This function can be generalized such that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Larger values denote higher degrees of set membership. Such a function is called a membership function and the set defined by it is a fuzzy set. Thus the membership function μ_A by which a fuzzy set A is defined has the form $\mu_A : X \rightarrow [0,1]$. This mathematical formulation of a fuzzy set was introduced by L.A. Zadeh in 1965 [93] as a generalization of a crisp set. Fuzzy logic is a logic based on fuzzy set theory. Hence the well established isomorphisms between Boolean algebra, set theory and propositional logic can be extended in a natural way between fuzzy algebra, fuzzy set theory and fuzzy logic. A result in any one of these theories has a counterpart in each of the other two theories (chapter 2 and chapter 6 of [36]).

Initially, fuzzy set theory was proposed by L.A. Zadeh as a means of representing mathematically any imprecise (or) vague system of information in the real world and for the purposes of developing expert systems and soft computing. In the last thirty years Lowen [44], Rosenfeld [79], Dubois and Prade [16], Zimmermann [100], Bell [3], Meenakshi *et al.* [58], [59], [60], Kerre [28] and others have extended the ideas of fuzzy set theory to topology, algebra, Hilbert spaces, graphs, game theory, logic and computing. For pioneering work on the theory of fuzzy sets, see Zadeh [93]. For further development of theory and applications of fuzzy sets, the reader is referred to Zadeh [94], [95], Negoita and Ralescu [73], Kaufmann [27] and Klir and Folger [36]. For generalization of fuzzy sets to lattice fuzzy sets (or) L-fuzzy sets, one may refer [20].

By a fuzzy matrix, we mean a matrix over a fuzzy algebra. A Boolean matrix is a special case of a fuzzy matrix with entries from the set $\{0,1\}$. In practice, fuzzy matrices have proposed to represent fuzzy relations in a system based on fuzzy set theory. A fuzzy matrix can be interpreted as a binary fuzzy relation. There are many research papers, books, lecture notes and dissertations devoted to binary fuzzy relations and it is a well shaped and vital part of fuzzy mathematics [35, 42, 90]. Among the general references to studies on fuzzy relations we would mention Zadeh [94], Kaufmann [27] and Rosenfeld [80].

There exist many papers concerning the abstract characterization of the semigroup of fuzzy relations and its fuzzy subsemigroups. N. Kuroki [38, 39, 40, 41], introduced the concepts of fuzzy subsemigroups and various fuzzy ideals of a semigroup. The theories of fuzzy matrices and semirings have found a number of applications, see Wechler [89], Gaines and Kohout [18] and chapter 5 of Negoita and Ralescu [73]. For certain properties of semigroups of fuzzy matrices, see J.B. Kim [29, 30, 31].

However, relatively little work has been done on fuzzy matrices. Recently, fuzzy matrices over the fuzzy algebra $\mathcal{F} = [0,1]$ has attracted the attention of several authors and they have developed the theory to some extent. In 1977, Thomason [87] initiated the study on convergence of powers of fuzzy matrices. Since then, many authors have discussed on the subject [9,86,92] with wide applications in fuzzy dynamical systems. The notion of fuzzy relational equations based upon the max–min composition was first proposed and investigated by Sanchez [82]. Klir and Folger [36], T.J. Ross [81], Gupta *et al.* [22], Wang and Chang [88], Cornelius and Leondes [13] and others have devoted a chapter (or) two to fuzzy matrices as a computational tool for analysis and synthesis of the behaviour of complex natural and artificial systems. In [32], K.H. Kim has presented a survey on Boolean matrices and the theory related to them.

In this book, we will be dealing with $\mathcal{F}_{mn}$, the set of all $m \times n$ fuzzy matrices over the fuzzy algebra $\mathcal{F} = [0,1]$ under the max–min operations and with the usual ordering of real numbers. In short, $\mathcal{F}_n$ denotes the set of all $n \times n$ matrices over $\mathcal{F}$. We usually suppress the dot '$\cdot$' of $a \cdot b$ and simply write ab. Hereafter, by a vector (or) matrix, we shall mean a vector (or) matrix over the fuzzy algebra $\mathcal{F} = [0,1]$ unless stated otherwise.

1.1 FUZZY VECTORS

In order of developments, the next structure we take up is a fuzzy vector. A fuzzy vector is an n-tuple of elements from a fuzzy algebra. Such n-tuples can be added, multiplied by elements of the fuzzy algebra (or) operated upon by matrices over a fuzzy algebra. Since fuzzy matrices over the max–min fuzzy algebra $\mathcal{F} = [0,1]$ is of prime importance in this book, we confine to fuzzy vectors over $\mathcal{F}$. Most of the material in this section, for any commutative semiring may be found in K.H. Kim & Roush [34]. Further the fuzzy algebra $\mathcal{F} = [0, 1]$ is also a commutative semiring with additive and multiplicative identities 0 and 1 respectively.

Definition 1.1.1

Let V_n denote the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ over $\mathcal{F}$. An element of V_n is called a fuzzy vector of dimension n. The operations $(+, \cdot)$ are defined on V_n as follows:

For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in V_n,

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

and $ax = (ax_1, ax_2, \ldots, ax_n)$ for $a \in \mathcal{F}$.

The system V_n together with these operations of componentwise addition and fuzzy multiplication is called a fuzzy vector space (or) a vector space over $\mathcal{F}$, as the scalars are restricted to $\mathcal{F}$. This will be sufficient for our purpose. Let O denote the zero vector $(0,0\ldots0)$.

Remark 1.1.1 V_n can be made into a fuzzy algebra by defining two operations $+, \cdot$ on it as follows:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \text{ and}$$

$$(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n) = (x_1 y_1, x_2 y_2, \ldots, x_n y_n)$$

for all $x_i, y_i \in \mathcal{F}$. For a constant vector $(a, a, \ldots, a)$ with $a \in \mathcal{F}$, the fuzzy multiplication $ax = (a, a, \ldots, a) \cdot (x_1, x_2, \ldots, x_n)$ is the multiple of the constant vectors a and $x \in V_n$.

Definition 1.1.2

Let $V^n = \{x^T / x \in V_n\}$ where x^T is the transpose of the vector x. For $u, v \in V^n$, $a \in \mathcal{F}$, define $av = (a^T v^T)$; $u + v = (u^T + v^T)^T$. Then V^n is a fuzzy vector space. If we write an element of V_n as a $1 \times n$ matrix, it is called a row vector. The elements of V^n are column vectors. For any result about V_n there exists a corresponding result about V^n. Thus V^n is isomorphic to V_n as a fuzzy algebra.

Definition 1.1.3

A subspace of V_n is a subset W of V_n such that $O \in W$ and for $x, y \in W$, $x + y \in W$.

Example 1.1.1 The set B_n of all n-tuples $(a_1, a_2, \ldots, a_n)$ over the two elements Boolean algebra $B = \{0, 1\}$ is a Boolean vector space of dimension $n \cdot B_n$ is a subspace of V_n.

Example 1.1.2 $W = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1), (1,1,1)\}$ is a subspace of V_3.

Definition 1.1.4

A linear combination of elements of set S is a finite sum $\Sigma a_i x_i$ where $x_i \in S$ and $a_i \in \mathcal{F}$. The set of all linear combinations of elements of S is called the span of S, denoted as $<S>$.

Definition 1.1.5

For subsets S, W of V_n if $<S> = W$, then S is called a spanning set (or) set of generators for W. If W is a subspace of V_n, then $<W> = W$.

Definition 1.1.6

A basis for a subspace W of V_n is a minimal spanning set for W.

Example 1.1.3 $W = \{(x, x) / x \in \mathcal{F}\}$ is a subspace of $\mathcal{F} \times \mathcal{F}$. The singleton set $S = \{(1, 1)\}$ is the minimal spanning set for W, since every element $(x, x) = x(1, 1)$ with $x \in \mathcal{F}$ is a linear combination of (1, 1) in S, S is a basis for W.

Definition 1.1.7

A set S of vectors over a fuzzy algebra $\mathcal{F}$ is independent if and only if each element of S is not a linear combination of other elements of S, that is, no element $v \in S$ is a linear combination of $S \setminus \{v\}$ (where $\setminus$ denotes the set theoretic difference).

Definition 1.1.8

A set S of vectors over $\mathcal{F}$ is dependent if it is not a linearly independent set.

Proposition 1.1.1

i. The set consisting of the zero vector alone is linearly dependent.

ii. If $X \subset Y$ and if X is linearly dependent then so is Y.

iii. If $X \subset Y$ and if Y is linearly independent then so is X.

Example 1.1.4 The set of vectors {(0.5, 1), (1, 0.6), (0.7, 0.9)} is a dependent set, since (0.7, 0.9) = 0.9 (0.5, 1) + 0.7 (1, 0.6).

Example 1.1.5 Let $S = \{a_1, a_2, a_3, a_4\}$ be a subset of V_3 given by $a_1 = (0.6, 0.3, 0.6)$, $a_2 = (0.4, 0.5, 0.5)$, $a_3 = (0.5, 0.6, 0.5)$ and $a_4 = (0.8, 0.6, 0.7)$. We claim that S is an independent set. For if a_1 is a linear combination of a_2, a_3 and a_4, then $a_1 = \alpha a_2 + \beta a_3 + \gamma a_4$ for $\alpha, \beta, \gamma \in \mathcal{F}$. Equating the corresponding coefficients on both sides we get,

$$0.6 = \alpha(0.4) + \beta(0.5) + \gamma(0.8) \Rightarrow \gamma = 0.6$$
$$0.3 = \alpha(0.5) + \beta(0.6) + \gamma(0.6).$$

Substituting $\gamma = 0.6$, we get,

$$0.3 = \alpha(0.5) + \beta(0.6) + 0.6,$$

that is $0.3 = 0.6$ for all $\alpha, \beta \in \mathcal{F}$ which is not possible.

Hence S is an independent subset of V_3.