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Symplectic Topology and Floer Homology

Volume 2: Floer Homology and its Applications

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Symplectic Topology and Floer Homology

Volume 2

Published in two volumes, this is the first book to provide a thorough and systematic explanation of symplectic topology, and the analytical details and techniques used in applying the machinery arising from Floer theory as a whole.

Volume 1 covers the basic materials of Hamiltonian dynamics and symplectic geometry and the analytic foundations of Gromov's pseudoholomorphic curve theory. One novel aspect of this treatment is the uniform treatment of both closed and open cases and a complete proof of the boundary regularity theorem of weak solutions of pseudoholomorphic curves with totally real boundary conditions. Volume 2 provides a comprehensive introduction to both Hamiltonian Floer theory and Lagrangian Floer theory, including many examples of their applications to various problems in symplectic topology.

Symplectic Topology and Floer Homology is a comprehensive resource suitable for experts and newcomers alike.

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Contents of Volume 1

Preface

This is a two-volume series of monographs. This series provides a self-contained exposition of basic Floer homology in both open and closed string contexts, and systematic applications to problems in Hamiltonian dynamics and symplectic topology. The basic objects of study in these two volumes are the geometry of Lagrangian submanifolds and the dynamics of Hamiltonian diffeomorphisms and their interplay in symplectic topology.

The classical Darboux theorem in symplectic geometry reveals the *flexibility* of the group of symplectic transformations. On the other hand, Gromov and Eliashberg's celebrated theorem (El87) reveals the subtle *rigidity* of symplectic transformations: *the subgroup* $\mathrm{Symp}(M, \omega)$ *consisting of symplectomorphisms is closed in* $\mathrm{Diff}(M)$ *with respect to the* C^0 *topology.* This demonstrates that the study of symplectic topology is subtle and interesting. Eliashberg's theorem relies on a version of the non-squeezing theorem, such as the one proved by Gromov (Gr85) using the machinery of pseudoholomorphic curves. Besides Eliashberg's original combinatorial proof of this non-squeezing result, there is another proof given by Ekeland and Hofer (EkH89) using the classical variational approach of Hamiltonian systems. The interplay between these two facets of symplectic geometry, namely the analysis of pseudoholomorphic curves and Hamiltonian dynamics, has been the main driving force in the development of symplectic topology since Floer's pioneering work on his semi-infinite dimensional homology theory, which we now call *Floer homology* theory.

Hamilton's equation $\dot{x} = X_H(t, x)$ arises in Hamiltonian mechanics and the study of its dynamics has been a fundamental theme of investigation in physics since the time of Lagrange, Hamilton, Jacobi, Poincaré and others. Many mathematical tools have been developed in the course of understanding its dynamics and finding explicit solutions of the equation. One crucial tool for the study of the questions is the *least action principle*: a solution of Hamilton's equation corresponds to a critical point of some action functional. In this variational principle, there are two most important boundary conditions considered for the equation $\dot{x} = X_H(t,x)$ on a general symplectic manifold: one is the *periodic* boundary condition $\gamma(0) = \gamma(1)$, and the other is the *Lagrangian* boundary condition $\gamma(0) \in L_0$, $\gamma(1) \in L_1$ for a given pair (L_0, L_1) of two Lagrangian submanifolds. A submanifold $i : L \hookrightarrow (M,\omega)$ is called Lagrangian if $i^*\omega = 0$ and $\dim L = \frac{1}{2}\dim M$. This replaces the *two-point* boundary condition in classical mechanics.

A diffeomorphism ϕ of a symplectic manifold (M,ω) is called a Hamiltonian diffeomorphism if ϕ is the time-one map of $\dot{x} = X_H(t,x)$ for some (time-dependent) Hamiltonian H. The set of such diffeomorphisms is denoted by $\mathrm{Ham}(M,\omega)$. It forms a subgroup of $\mathrm{Symp}(M,\omega)$. However, in the author's opinion, it is purely a historical accident that Hamiltonian diffeomorphisms are studied because the definition of $\mathrm{Ham}(M,\omega)$ is not a priori natural. For example, it is not a structure group of any geometric structure associated with the smooth manifold M (or at least not of any structure known as yet), unlike the case of $\mathrm{Symp}(M,\omega)$, which is the automorphism group of the symplectic structure ω. Mathematicians' interest in $\mathrm{Ham}(M,\omega)$ is largely motivated by the celebrated conjecture of Arnol'd (Ar65), and Floer homology was invented by Floer (Fl88b, Fl89b) in his attempt to prove this conjecture.

Since the advent of Floer homology in the late 1980s, it has played a fundamental role in the development of symplectic topology. (There is also the parallel notion in lower-dimensional topology which is not touched upon in these two volumes. We recommend for interested readers Floer's original article (Fl88c) and the masterpiece of Kronheimer and Mrowka (KM07) in this respect.) Owing to the many technicalities involved in its rigorous definition, especially in the case of Floer homology of Lagrangian intersections (or in the context of 'open string'), the subject has been quite inaccessible to beginning graduate students and researchers coming from other areas of mathematics. This is partly because there is no existing literature that systematically explains the problems of symplectic topology, the analytical details and the techniques involved in applying the machinery embedded in the Floer theory as a whole. In the meantime, Fukaya's categorification of Floer homology, i.e., his introduction of an A_∞ category into symplectic geometry (now called the Fukaya category), and Kontsevich's homological mirror symmetry proposal, followed by the development of open string theory of D branes in physics, have greatly enhanced the Floer theory and attracted much attention from other mathematicians and physicists as well as the traditional symplectic geometers and topologists. In addition, there has also been considerable research into applications of symplectic ideas to various problems in (area-preserving) dynamical systems in two dimensions.

Our hope in writing these two volumes is to remedy the current difficulties to some extent. To achieve this goal, we focus more on the foundational materials of Floer theory and its applications to various problems arising in symplectic topology, with which the author is more familiar, and attempt to provide complete analytic details assuming the reader's knowledge of basic elliptic theory of (first-order) partial differential equations, second-year graduate differential geometry and first-year algebraic topology. In addition, we also try to motivate various constructions appearing in Floer theory from the historical context of the classical Lagrange–Hamilton variational principle and Hamiltonian mechanics. The choice of topics included in the book is somewhat biased, partly due to the limitations of the author's knowledge and confidence level, and also due to his attempt to avoid too much overlap with the existing literature on symplectic topology. We would like to particularly cite the following three monographs among others and compare these two volumes with them:

(1) *J-Holomorphic Curves and Symplectic Topology*, McDuff, D., Salamon, D., 2004.
(2) *Fukaya Categories and Picard–Lefschetz Theory*, Seidel, P., 2008.
(3) *Lagrangian Intersection Floer Theory: Anomaly and Obstruction*, volumes I & II, Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K, 2009.

(There is another more recent monograph by Audin and Damian (AD14), which was originally written in French and then translated into English.)

First of all, Parts 2 and 3 of these two volumes could be regarded as the prerequisite for graduate students or post-docs to read the book (3) (FOOO09) in that the off-shell setting of Lagrangian Floer theory in Volume 2 presumes the presence of non-trivial instantons, or non-constant holomorphic discs or spheres. However, we largely limit ourselves to the monotone case and avoid the full-fledged obstruction–deformation theory of Floer homology which would inevitably involve the theory of A_∞-structures and the abstract perturbation theory of virtual moduli technique such as the Kuranishi structure, which is beyond the scope of these two volumes. Luckily, the books (2) (Se08) and (3) (FOOO09) cover this important aspect of the theory, so we strongly encourage readers to consult them. We also largely avoid any extensive discussion on the Floer theory of exact Lagrangian submanifolds, except for the cotangent bundle case, because Seidel's book (Se08) presents an extensive study of the Floer theory and the Fukaya category in the context of exact symplectic geometry to which we cannot add anything. There is much overlap of the materials in Part 2 on the basic pseudoholomorphic curve theory with Chapters 1–6 of the book by McDuff and Salamon (1) (MSa04). However, our exposition of the materials is quite different from that of (MSa04). For example, from the beginning, we deal with pseudoholomorphic curves of arbitrary genus and with a boundary and unify the treatment of both closed and open cases, e.g., in the regularity theory of weak solutions and in the removal singularity theorem. Also we discuss the transversality issue after that of compactness, which seems to be more appropriate for accommodating the techniques of Kuranishi structure and abstract perturbation theory when the readers want to go beyond the semi-positive case. There are also two other points that we are particularly keen about in our exposition of pseudoholomorphic curve theory. One is to make the relevant geometric analysis resemble the style of the more standard geometric analysis in Riemannian geometry, emphasizing the tensor calculations via the canonical connection associated with the almost-Kähler property whenever possible. In this way we derive the relevant $W^{k,p}$-coercive estimates, especially an optimal $W^{2,2}$-estimate with Neumann boundary condition, by pure tensor calculations and an application of the Weitzenböck formula. The other is to make the deformation theory of pseudoholomorphic curves resemble that of holomorphic curves on (integrable) Kähler manifolds. We hope that this style of exposition will widen the readership beyond the traditional symplectic geometers to graduate students and researchers from other areas of mathematics and enable them to more easily access important developments in symplectic topology and related areas.

Now comes a brief outline of the contents of each part of the two volumes. The first volume consists of Parts 1 and 2.

Part 1 gives an introduction to symplectic geometry starting from the classical variational principle of Lagrange and Hamilton in classical mechanics and introduces the main concepts in symplectic geometry, i.e., Lagrangian submanifolds, Hamiltonian diffeomorphisms and symplectic fibrations. It also introduces Hofer's geometry of Hamiltonian diffeomorphisms. Then the part ends with the proof of the Gromov–Eliashberg C^0-rigidity theorem (El87) and the introduction to continuous Hamiltonian dynamics and the concept of Hamiltonian homeomorphisms introduced by Müller and the present author (OhM07).

Part 2 provides a mostly self-contained exposition of the analysis of pseudoholomorphic curves and their moduli spaces. We attempt to provide the optimal form of a-priori elliptic estimates for the nonlinear Cauchy–Riemann operator $\overline{\partial}_J$ *in the off-shell setting*. For this purpose, we emphasize our usage of the canonical connection of the almost-Kähler manifold (M, ω, J). Another novelty of our treatment of the analysis is a complete proof of the boundary regularity theorem of weak solutions (in the sense of Ye (Ye94)) of J-holomorphic curves with totally real boundary conditions. As far as we know, this regularity proof has not been given before in the existing literature. We also give a complete proof of compactness of the stable map moduli space following the approach taken by Fukaya and Ono (FOn99). The part ends with an explanation of how compactness–cobordism analysis of the moduli space of (perturbed) pseudoholomorphic curves combined with a bit of symplectic topological data give rise to the proofs of two basic theorems in symplectic topology; Gromov's non-squeezing theorem and the nondegeneracy of Hofer's norm on $\mathrm{Ham}(M, \omega)$ (for tame symplectic manifolds).

The second volume consists of Parts 3 and 4. Part 3 gives an introduction to Lagrangian Floer homology restricted to the special cases of monotone Lagrangian submanifolds. It starts with an overview of Lagrangian intersection Floer homology on cotangent bundles and introduces all the main objects of study that enter into the recent Lagrangian intersection Floer theory without delving too much into the technical details. Then it explains the compactification of Floer moduli spaces, the details of which are often murky in the literature. The part ends with the construction of a spectral sequence, a study of Maslov class obstruction to displaceable Lagrangian submanifolds and Polterovich's theorem on the Hofer diameter of $\mathrm{Ham}(S^2)$.

Part 4 introduces Hamiltonian Floer homology and explains the complete construction of spectral invariants and various applications. The applications include construction of the spectral norm, Usher's proofs of the minimality conjecture in the Hofer geometry and the optimal energy–capacity inequality. In particular, this part contains a complete self-contained exposition of the Entov–Polterovich construction of spectral quasimorphisms and the associated symplectic quasi-states. The part ends with further discussion of topological Hamiltonian flows and their relation to the geometry of area-preserving homeomorphisms in two dimensions.

The prerequisites for the reading of these two volumes vary part by part. A standard first-year graduate differentiable manifold course together with a little bit of knowledge on the theory of fiber bundles should be enough for Part 1. However, Part 2, the most technical part of the book, which deals with the general theory of pseudoholomorphic curves, the moduli spaces thereof and their stable map compactifications, assumes readers' knowledge of the basic language of Riemannian geometry (e.g., that of Volumes 1 and 2 of Spivak (Spi79)), basic functional analysis (e.g., Sobolev embedding and Reillich compactness and others), elliptic (first-order) partial differential equations and first-year algebraic topology. The materials in Parts 3 and 4, which deal with the main topics of Floer homology both in the open and in the closed string context, rely on the materials of Parts 1 and 2 and should be readable on their own. Those who are already familiar with basic symplectic geometry and analysis of pseudoholomorphic curves should be able to read Parts 3 and 4 immediately. This book can be used as a graduate textbook for the introduction to Gromov and Floer's analytic approach to modern symplectic topology. Readers who would like to learn more about various deeper aspects of symplectic topology and mirror symmetry are strongly encouraged to read the books (1)–(3) mentioned above in addition, depending on their interest.

The author would like to end this preface by recalling his personal experience and perspective, which might not be shared by others, but which he hopes may help readers to see how the author came up with the current shape of these two volumes. The concept of symplectic topology emerged from Eliashberg and Gromov's celebrated symplectic rigidity theorem. Eliashberg's original proof was based on the existence of some C^0-type invariant of the symplectic diffeomorphism which measures the size of domains in the symplectic vector space. The existence of such an invariant was first established by Gromov as a corollary of his fundamental *non-squeezing theorem* that was proven by using the analytical method of pseudoholomorphic curves. With the advent of the method of pseudoholomorphic curves developed by Gromov and Floer's subsequent invention of elliptic Morse theory that resulted in Floer homology, the landscape of symplectic geometry changed drastically. Many previously intractable problems in symplectic geometry were solved by the techniques of pseudoholomorphic curves, and the concept of symplectic topology gradually began to take shape.

There are two main factors determining how the author shaped the structure of the present book. The first concerns how the analytical materials are treated in Volume 1. The difficulty, or the excitement, associated with the method of pseudoholomorphic curves at the time of its appearance was that it involves a mathematical discipline of a nature very different from the type of mathematics employed by the mainstream symplectic geometers at that time. As a result the author feels that it created some discontinuity between the symplectic geometry before and after Gromov's paper appeared, and the analysis presented was given quite differently from how such matters are normally treated by geometric analysts of Riemannian geometry. For example, the usage of tensor calculations is not emphasized as much as in Riemannian geometry. Besides, in the author's personal experience, there were two stumbling blocks hindering getting into the Gromov–Witten–Floer theory as a graduate student and as a beginning researcher working in the area of symplectic topology. The first was the need to get rid of some phobia towards the abstract algebraic geometric materials like the Deligne–Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves, and the other was the need to absorb the large amount of analytical materials that enter into the study of moduli spaces of pseudoholomorphic maps from the original sources in the literature of the relevant mathematics whose details are often too sketchy. It turns out that many of these details are in some sense standard in the point of view of geometric analysts and can be treated in a more effective way using the standard tensorial methods of Riemannian geometry.

The second concerns how the Floer theory is presented in the book. In the author's personal experience, it seems to be most effective to learn the Floer theory both in the closed and in the open string context simultaneously. Very often problems on the Hamiltonian dynamics are solved via the corresponding problems on the geometry of Lagrangian intersections. For this reason, the author presents the Floer theory of the closed and the open string context at the same time. While the technical analytic details of pseudoholomorphic curves are essentially the same for both closed and open string contexts, the relevant geometries of the moduli space of pseudoholomorphic curves are different for the closed case and the open case of Riemann surfaces. This difference makes the Floer theory of Lagrangian intersection very different from that of Hamiltonian fixed points.

Acknowledgments

This book owes a great debt to many people whose invaluable help cannot be over-emphasized. The project of writing these two volumes started when I offered a three-semester-long course on symplectic geometry and pseudoholomorphic curves at the University of Wisconsin–Madison in the years 2002–2004 and another quarter course on Lagrangian Floer homology while I was on sabbatical leave at Stanford University in the year 2004–2005.

I learned most of the materials on basic symplectic geometry from Alan Weinstein as his graduate student in Berkeley. Especially, the majority of the materials in Part 1 of Volume 1 are based on the lecture notes I had taken for his year-long course on symplectic geometry in 1987–1988. One could easily see the widespread influence of his mathematics throughout Part 1. I would like to take this chance to sincerely thank him for his invaluable support and encouragement throughout my graduate study.

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I thank Michael Entov and Leonid Polterovich for their astounding construction of symplectic quasi-states and quasimorphisms, which I believe starts to unveil the true symplectic nature of the analytical construction of Floer homology and spectral invariants. I also thank Claude Viterbo, Michael Usher, Lev Buhovsky and Soban Seyfaddini for their important mathematical works, which are directly relevant to the author's more recent research on spectral invariants and topological Hamiltonian dynamics and which the author very much appreciates.

I also thank various institutions in which I spent some time during the writing of the book. Special thanks go to the University of Wisconsin–Madison, where the majority of the writing of the present volumes was carried out while I was a member of the faculty there. I also thank the Korea Institute for Advanced Study (KIAS), the National Institute of Mathematical Sciences (NIMS) and Seoul National University, which provided financial support and an excellent research environment during my stay there.

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List of conventions

We follow the conventions of (Oh05c, Oh09a, Oh10) for the definition of Hamiltonian vector fields and action functionals and others appearing in the Hamiltonian Floer theory and in the construction of spectral invariants and Entov–Polterovich Calabi quasimorphisms. They are different from, e.g., those used in (EnP03, EnP06, EnP09) in one way or another.

(1) The canonical symplectic form ω_0 on the cotangent bundle T^*N is given by

$$\omega_0 = -d\Theta = \sum_{i=1}^{n} dq^i \wedge dp_i,$$