



Donald T. Greenwood

# **Advanced Dynamics**

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**Donald T. Greenwood**

University of Michigan



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## Advanced Dynamics

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*Advanced Dynamics* is a broad and detailed description of the analytical tools of dynamics as used in mechanical and aerospace engineering. The strengths and weaknesses of various approaches are discussed, and particular emphasis is placed on learning through problem solving.

The book begins with a thorough review of vectorial dynamics and goes on to cover Lagrange's and Hamilton's equations as well as less familiar topics such as impulse response, and differential forms and integrability. Techniques are described that provide a considerable improvement in computational efficiency over the standard classical methods, especially when applied to complex dynamical systems. The treatment of numerical analysis includes discussions of numerical stability and constraint stabilization. Many worked examples and homework problems are provided. The book is intended for use in graduate courses on dynamics, and will also appeal to researchers in mechanical and aerospace engineering.

**Donald T. Greenwood** received his Ph.D. from the California Institute of Technology, and is a Professor Emeritus of aerospace engineering at the University of Michigan, Ann Arbor. Before joining the faculty at Michigan he worked for the Lockheed Aircraft Corporation, and has also held visiting positions at the University of Arizona, the University of California, San Diego, and ETH Zurich. He is the author of two previous books on dynamics.

# Preface

This is a dynamics textbook for graduate students, written at a moderately advanced level. Its principal aim is to present the dynamics of particles and rigid bodies in some breadth, with examples illustrating the strengths and weaknesses of the various methods of dynamical analysis. The scope of the dynamical theory includes both vectorial and analytical methods. There is some emphasis on systems of great generality, that is, systems which may have nonholonomic constraints and whose motion may be expressed in terms of quasi-velocities. Geometrical approaches such as the use of surfaces in  $n$ -dimensional configuration and velocity spaces are used to illustrate the nature of holonomic and nonholonomic constraints. Impulsive response methods are discussed at some length.

Some of the material presented here was originally included in a graduate course in computational dynamics at the University of Michigan. The ordering of the chapters, with the chapters on dynamical theory presented first followed by the single chapter on numerical methods, is such that the degree of emphasis one chooses to place on the latter is optional. Numerical computation methods may be introduced at any point, or may be omitted entirely.

The first chapter presents in some detail the familiar principles of Newtonian or vectorial dynamics, including discussions of constraints, virtual work, and the use of energy and momentum principles. There is also an introduction to less familiar topics such as differential forms, integrability, and the basic theory of impulsive response.

Chapter 2 introduces methods of analytical dynamics as represented by Lagrange's and Hamilton's equations. The derivation of these equations begins with the Lagrangian form of d'Alembert's principle, a common starting point for obtaining many of the principal forms of dynamical equations of motion. There are discussions of ignorable coordinates, the Routhian method, and the use of integrals of the motion. Frictional and gyroscopic forces are studied, and further material is presented on impulsive systems.

Chapter 3 is concerned with the kinematics and dynamics of rigid body motion. Dyadic and matrix notations are introduced. Euler parameters and axis-and-angle variables are used extensively in representing rigid body orientations in addition to the more familiar Euler angles. This chapter also includes material on constrained impulsive response and input-output methods.

The theoretical development presented in the first three chapters is used as background for the derivations of Chapter 4. Here we present several differential methods which have the advantages of simplicity and computational efficiency over the usual Lagrangian methods

in the analysis of general constrained systems or for systems described in terms of quasi-velocities. These methods result in a minimum set of dynamical equations which are computationally efficient. Many examples are included in order to compare and explain the various approaches. This chapter also presents detailed discussions of constraints and energy rates by using velocity space concepts.

Chapter 5 begins with a derivation of Hamilton's principle in its holonomic and nonholonomic forms. Stationarity questions are discussed. Transpositional relations are introduced and there follows a further discussion of integrability including Frobenius' theorem. The central equation and its explicit transpositional form are presented. There is a comparison of integral methods by means of examples.

Chapter 6 presents some basic principles of numerical analysis and explains the use of integration algorithms in the numerical solution of differential equations. For the most part, explicit algorithms such as the Runge–Kutta and predictor–corrector methods are considered. There is an analysis of numerical stability of the integration methods, primarily by solving the appropriate difference equations, but frequency response methods are also used. The last portion of the chapter considers methods of representing kinematic constraints. The one-step method of constraint stabilization is introduced and its advantages over standard methods are explained. There is a discussion of the use of energy and momentum constraints as a means of improving the accuracy of numerical computations.

A principal objective of this book is to improve the problem-solving skills of each student. Problem solving should include not only a proper formulation and choice of variables, but also a directness of approach which avoids unnecessary steps. This requires that the student repeatedly attempt the solution of problems which may be kinematically complex and which involve the application of several dynamical principles. The problems presented here usually have several parts that require more than the derivation of the equations of motion for a given system. Thus, insight is needed concerning other dynamical characteristics. Because of the rather broad array of possible approaches presented here, and due in part to the generally demanding problems, a conscientious student can attain a real perspective of the subject of dynamics and a competence in the application of its principles.

Finally, I would like to acknowledge the helpful discussions with Professor J. G. Papastavridis of Georgia Tech concerning the material of Chapters 4 and 5, and with Professor R. M. Howe of the University of Michigan concerning portions of Chapter 6.

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# 1 Introduction to particle dynamics

In the study of dynamics at an advanced level, it is important to consider many approaches and points of view in order that one may attain a broad theoretical perspective of the subject. As we proceed we shall emphasize those methods which are particularly effective in the analysis of relatively difficult problems in dynamics. At this point, however, it is well to review some of the basic principles in the dynamical analysis of systems of particles. In the process, the kinematics of particle motion will be reviewed, and many of the notational conventions will be established.

## 1.1 Particle motion

### The laws of motion for a particle

Let us consider Newton's three laws of motion which were published in 1687 in his *Principia*. They can be stated as follows:

- I. Every body continues in its state of rest, or of uniform motion in a straight line, unless compelled to change that state by forces acting upon it.
- II. The time rate of change of linear momentum of a body is proportional to the force acting upon it and occurs in the direction in which the force acts.
- III. To every action there is an equal and opposite reaction; that is, the mutual forces of two bodies acting upon each other are equal in magnitude and opposite in direction.

In the dynamical analysis of a system of particles using Newton's laws, we can interpret the word "body" to mean a particle, that is, a certain fixed mass concentrated at a point. The first two of Newton's laws, as applied to a particle, can be summarized by the *law of motion*:

$$\mathbf{F} = m\mathbf{a} \tag{1.1}$$

Here  $\mathbf{F}$  is the total force applied to the particle of mass  $m$  and it includes both direct contact forces and field forces such as gravity or electromagnetic forces. The acceleration  $\mathbf{a}$  of the particle must be measured relative to an *inertial* or *Newtonian* frame of reference. An example of an inertial frame is an  $xyz$  set of axes which is not rotating relative to the "fixed"

stars and has its origin at the center of mass of the solar system. Any other reference frame which is not rotating but is translating at a constant rate relative to an inertial frame is itself an inertial frame. Thus, there are infinitely many inertial frames, all with constant translational velocities relative to the others. Because the relative velocities are constant, the acceleration of a given particle is the same relative to any inertial frame. The force  $\mathbf{F}$  and mass  $m$  are also the same in all inertial frames, so Newton's law of motion is identical relative to all inertial frames.

Newton's third law, the law of action and reaction, has a corollary assumption that the interaction forces between any two particles are directed along the straight line connecting the particles. Thus we have the *law of action and reaction*:

*When two particles exert forces on each other, these interaction forces are equal in magnitude, opposite in sense, and are directed along the straight line joining the particles.*

The *collinearity* of the interaction forces applies to all mechanical and gravitational forces. It does not apply, however, to interactions between moving electrically charged particles for which the interaction forces are equal and opposite but not necessarily collinear. Systems of this sort will not be studied here.

An alternative form of the equation of motion of a particle is

$$\mathbf{F} = \dot{\mathbf{p}} \quad (1.2)$$

where the *linear momentum* of the particle is

$$\mathbf{p} = m\mathbf{v} \quad (1.3)$$

and  $\mathbf{v}$  is the particle velocity relative to an inertial frame.

## Kinematics of particle motion

The application of Newton's laws of motion to a particle requires that an expression can be found for the acceleration of the particle relative to an inertial frame. For example, the position vector of a particle relative to a *fixed* Cartesian frame might be expressed as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (1.4)$$

where  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  are *unit vectors*, that is, vectors of unit magnitude which have the directions of the positive  $x$ ,  $y$ , and  $z$  axes, respectively. When unit vectors are used to specify a vector in 3-space, the three unit vectors are always linearly independent and are nearly always mutually perpendicular. The velocity of the given particle is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \quad (1.5)$$

and its acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \quad (1.6)$$

relative to the inertial frame.

A force  $\mathbf{F}$  applied to the particle may be described in a similar manner.

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \quad (1.7)$$

where  $(F_x, F_y, F_z)$  are the *scalar components* of  $\mathbf{F}$ . In general, the force components can be functions of position, velocity, and time, but often they are much simpler.

If one writes Newton's law of motion, (1.1), in terms of the Cartesian unit vectors, and then equates the scalar coefficients of each unit vector on the two sides of the equation, one obtains

$$\begin{aligned} F_x &= m\ddot{x} \\ F_y &= m\ddot{y} \\ F_z &= m\ddot{z} \end{aligned} \quad (1.8)$$

These three scalar equations are equivalent to the single vector equation. In general, the scalar equations are coupled through the expressions for the force components. Furthermore, the differential equations are often nonlinear and are not susceptible to a complete analytic solution. In this case, one can turn to numerical integration on a digital computer to obtain the complete solution. On the other hand, one can often use energy or momentum methods to obtain important characteristics of the motion without having the complete solution.

The calculation of a particle acceleration relative to an inertial Cartesian frame is straightforward because the unit vectors  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$  are fixed in direction. It turns out, however, that because of system geometry it is sometimes more convenient to use unit vectors that are not fixed. For example, the position, velocity, and acceleration of a particle moving along a circular path are conveniently expressed using radial and tangential unit vectors which change direction with position.

As a more general example, suppose that an arbitrary vector  $\mathbf{A}$  is given by

$$\mathbf{A} = A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{e}_3 \quad (1.9)$$

where the unit vectors  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  form a mutually orthogonal set such that  $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ . This unit vector triad changes its orientation with time. It rotates as a rigid body with an angular velocity  $\boldsymbol{\omega}$ , where the direction of  $\boldsymbol{\omega}$  is along the axis of rotation and the positive sense of  $\boldsymbol{\omega}$  is in accordance with the right-hand rule.

The first time derivative of  $\mathbf{A}$  is

$$\dot{\mathbf{A}} = \dot{A}_1 \mathbf{e}_1 + \dot{A}_2 \mathbf{e}_2 + \dot{A}_3 \mathbf{e}_3 + A_1 \dot{\mathbf{e}}_1 + A_2 \dot{\mathbf{e}}_2 + A_3 \dot{\mathbf{e}}_3 \quad (1.10)$$

where

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i \quad (i = 1, 2, 3) \quad (1.11)$$

Thus we obtain the important equation

$$\dot{\mathbf{A}} = (\dot{\mathbf{A}})_r + \boldsymbol{\omega} \times \mathbf{A} \quad (1.12)$$

Here  $\dot{\mathbf{A}}$  is the time rate of change of  $\mathbf{A}$ , as measured in a nonrotating frame that is usually considered to also be inertial.  $(\dot{\mathbf{A}})_r$  is the derivative of  $\mathbf{A}$ , as measured in a rotating frame in

which the unit vectors are fixed. It is represented by the first three terms on the right-hand side of (1.10). The term  $\boldsymbol{\omega} \times \mathbf{A}$  is represented by the final three terms of (1.10). In detail, if the angular velocity of the rotating frame is

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3 \quad (1.13)$$

then

$$\begin{aligned} \dot{\mathbf{A}} = & (\dot{A}_1 + \omega_2 A_3 - \omega_3 A_2) \mathbf{e}_1 + (\dot{A}_2 + \omega_3 A_1 - \omega_1 A_3) \mathbf{e}_2 \\ & + (\dot{A}_3 + \omega_1 A_2 - \omega_2 A_1) \mathbf{e}_3 \end{aligned} \quad (1.14)$$

### Velocity and acceleration expressions for common coordinate systems

Let us apply the general equation (1.12) to some common coordinate systems associated with particle motion.

#### Cylindrical coordinates

Suppose that the position of a particle  $P$  is specified by the values of its cylindrical coordinates  $(r, \phi, z)$ . We see from Fig. 1.1 that the position vector  $\mathbf{r}$  is

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z \quad (1.15)$$

where we notice that  $r$  is not the magnitude of  $\mathbf{r}$ . The angular velocity of the  $\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z$  triad is

$$\boldsymbol{\omega} = \dot{\phi} \mathbf{e}_z \quad (1.16)$$

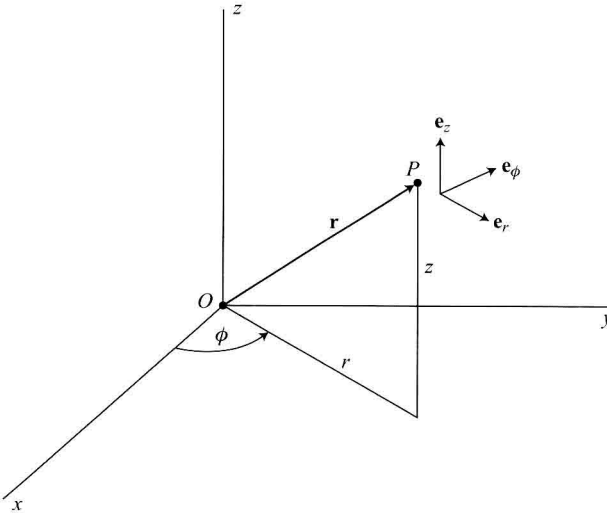


Figure 1.1.

so we find that  $\dot{\mathbf{e}}_z$  vanishes and

$$\dot{\mathbf{e}}_r = \boldsymbol{\omega} \times \mathbf{e}_r = \dot{\phi} \mathbf{e}_\phi \quad (1.17)$$

Thus, the velocity of the particle  $P$  is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r} \mathbf{e}_r + r \dot{\phi} \mathbf{e}_\phi + \dot{z} \mathbf{e}_z \quad (1.18)$$

Similarly, noting that

$$\dot{\mathbf{e}}_\phi = \boldsymbol{\omega} \times \mathbf{e}_\phi = -\dot{\phi} \mathbf{e}_r \quad (1.19)$$

we find that its acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r} - r\dot{\phi}^2) \mathbf{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi}) \mathbf{e}_\phi + \ddot{z} \mathbf{e}_z \quad (1.20)$$

If we restrict the motion such that  $\dot{z}$  and  $\ddot{z}$  are continuously equal to zero, we obtain the velocity and acceleration equations for plane motion using *polar coordinates*.

### Spherical coordinates

From Fig. 1.2 we see that the position of particle  $P$  is given by the spherical coordinates  $(r, \theta, \phi)$ . The position vector of the particle is simply

$$\mathbf{r} = r \mathbf{e}_r \quad (1.21)$$

The angular velocity of the  $\mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\phi$  triad is due to  $\dot{\theta}$  and  $\dot{\phi}$  and is equal to

$$\boldsymbol{\omega} = \dot{\phi} \cos \theta \mathbf{e}_r - \dot{\phi} \sin \theta \mathbf{e}_\theta + \dot{\theta} \mathbf{e}_\phi \quad (1.22)$$

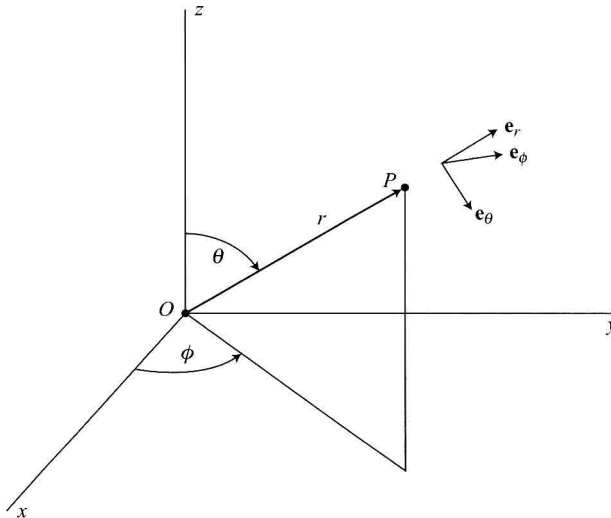


Figure 1.2.

We find that

$$\begin{aligned}\dot{\mathbf{e}}_r &= \boldsymbol{\omega} \times \mathbf{e}_r = \dot{\theta} \mathbf{e}_\theta + \dot{\phi} \sin \theta \mathbf{e}_\phi \\ \dot{\mathbf{e}}_\theta &= \boldsymbol{\omega} \times \mathbf{e}_\theta = -\dot{\theta} \mathbf{e}_r + \dot{\phi} \cos \theta \mathbf{e}_\phi \\ \dot{\mathbf{e}}_\phi &= \boldsymbol{\omega} \times \mathbf{e}_\phi = -\dot{\phi} \sin \theta \mathbf{e}_r - \dot{\phi} \cos \theta \mathbf{e}_\theta\end{aligned}\quad (1.23)$$

Then, upon differentiation of (1.21), we obtain the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta + r \dot{\phi} \sin \theta \mathbf{e}_\phi \quad (1.24)$$

A further differentiation yields the acceleration

$$\begin{aligned}\mathbf{a} = \dot{\mathbf{v}} &= (\ddot{r} - r \dot{\theta}^2 - r \dot{\phi}^2 \sin^2 \theta) \mathbf{e}_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta} - r \dot{\phi}^2 \sin \theta \cos \theta) \mathbf{e}_\theta \\ &\quad + (r \ddot{\phi} \sin \theta + 2 \dot{r} \dot{\phi} \sin \theta + 2 r \dot{\theta} \dot{\phi} \cos \theta) \mathbf{e}_\phi\end{aligned}\quad (1.25)$$

### Tangential and normal components

Suppose a particle  $P$  moves along a given path in three-dimensional space. The position of the particle is specified by the single coordinate  $s$ , measured from some reference point along the path, as shown in Fig. 1.3. It is convenient to use the three unit vectors ( $\mathbf{e}_t$ ,  $\mathbf{e}_n$ ,  $\mathbf{e}_b$ ) where  $\mathbf{e}_t$  is tangent to the path at  $P$ ,  $\mathbf{e}_n$  is normal to the path and points in the direction of the center of curvature  $C$ , and the binormal unit vector is

$$\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n \quad (1.26)$$

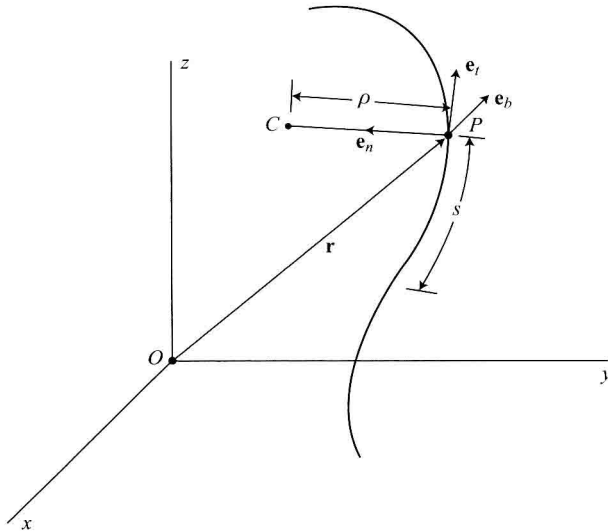


Figure 1.3.

The velocity of the particle is equal to its speed along its path, so

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{s}\mathbf{e}_t \quad (1.27)$$

If we consider motion along an infinitesimal arc of radius  $\rho$  surrounding  $P$ , we see that

$$\dot{\mathbf{e}}_t = \frac{\dot{s}}{\rho}\mathbf{e}_n \quad (1.28)$$

Thus, we find that the acceleration of the particle is

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{s}\mathbf{e}_t + \dot{s}\dot{\mathbf{e}}_t = \ddot{s}\mathbf{e}_t + \frac{\dot{s}^2}{\rho}\mathbf{e}_n \quad (1.29)$$

where  $\rho$  is the radius of curvature. Here  $\ddot{s}$  is the tangential acceleration and  $\dot{s}^2/\rho$  is the centripetal acceleration. The angular velocity of the unit vector triad is directly proportional to  $\dot{s}$ . It is

$$\boldsymbol{\omega} = \omega_t\mathbf{e}_t + \omega_b\mathbf{e}_b \quad (1.30)$$

where  $\omega_t$  and  $\omega_b$  are obtained from

$$\begin{aligned} \dot{\mathbf{e}}_t &= \omega_b\mathbf{e}_n = \frac{\dot{s}}{\rho}\mathbf{e}_n \\ \dot{\mathbf{e}}_b &= -\omega_t\mathbf{e}_n = \dot{s}\frac{d\mathbf{e}_b}{ds} \end{aligned} \quad (1.31)$$

Note that  $\omega_n = 0$  and also that  $d\mathbf{e}_b/ds$  represents the *torsion* of the curve.

## Relative motion and rotating frames

When one uses Newton's laws to describe the motion of a particle, the acceleration  $\mathbf{a}$  must be *absolute*, that is, it must be measured relative to an inertial frame. This acceleration, of course, is the same when measured with respect to any inertial frame. Sometimes the motion of a particle is known relative to a rotating and accelerating frame, and it is desired to find its absolute velocity and acceleration. In general, these calculations can be somewhat complicated, but for the special case in which the moving frame  $A$  is not rotating, the results are simple. The absolute velocity of a particle  $P$  is

$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{P/A} \quad (1.32)$$

where  $\mathbf{v}_A$  is the absolute velocity of any point on frame  $A$  and  $\mathbf{v}_{P/A}$  is the velocity of particle  $P$  relative to frame  $A$ , that is, the velocity recorded by cameras or other instruments fixed in frame  $A$  and moving with it. Similarly, the absolute acceleration of  $P$  is

$$\mathbf{a}_P = \mathbf{a}_A + \mathbf{a}_{P/A} \quad (1.33)$$

where we note again that the frame  $A$  is moving in pure translation.

Now consider the general case in which the moving  $xyz$  frame (Fig. 1.4) is translating and rotating arbitrarily. We wish to find the velocity and acceleration of a particle  $P$  relative

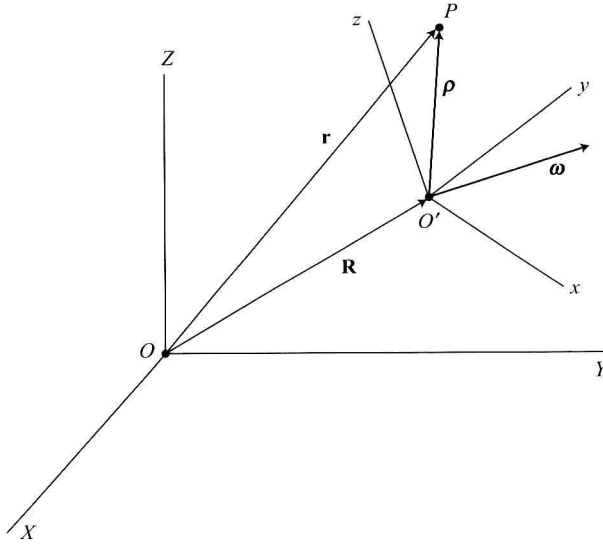


Figure 1.4.

to the inertial  $XYZ$  frame in terms of its motion with respect to the noninertial  $xyz$  frame. Let the origin  $O'$  of the  $xyz$  frame have a position vector  $\mathbf{R}$  relative to the origin  $O$  of the  $XYZ$  frame. The position of the particle  $P$  relative to  $O'$  is  $\boldsymbol{\rho}$ , so the position of  $P$  relative to  $XYZ$  is

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\rho} \quad (1.34)$$

The corresponding velocity is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\boldsymbol{\rho}} \quad (1.35)$$

Now let us use the basic equation (1.12) to express  $\dot{\boldsymbol{\rho}}$  in terms of the motion relative to the moving  $xyz$  frame. We obtain

$$\dot{\boldsymbol{\rho}} = (\dot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times \boldsymbol{\rho} \quad (1.36)$$

where  $\boldsymbol{\omega}$  is the angular velocity of the  $xyz$  frame and  $(\dot{\boldsymbol{\rho}})_r$  is the velocity of  $P$  relative to that frame. In detail,

$$\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (1.37)$$

and

$$(\dot{\boldsymbol{\rho}})_r = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \quad (1.38)$$

where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are unit vectors fixed in the  $xyz$  frame and rotating with it. From (1.35) and (1.36), the absolute velocity of  $P$  is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{R}} + (\dot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times \boldsymbol{\rho} \quad (1.39)$$