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Volume 1

Multiple Solutions of Boundary Value Problems

A Variational Approach

John R Graef • Lingju Kong

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To Frances, without whose help and support this work would never have been finished, and to our sons John, Chris, and Kevin.

Dedicated to my loving wife Zhen and son Michael.

Preface

Variational methods and their generalizations have proved to be a useful tool in proving the existence of solutions to a variety of boundary value problems for ordinary, impulsive, and partial differential equations as well as for difference equations. In this monograph, we look at how variational methods can be used in all these settings. In our first chapter, we gather the basic notions and fundamental theorems that will be applied in the remainder of this monograph. While many of these items are easily available in the literature, we gather them here both for the convenience of the reader and for the purpose of making this volume somewhat self contained. Subsequent chapters deal with Sturm-Liouville problems, multi-point boundary value problems, problems with impulses, partial differential equations, and difference equations. The chapters are presented in such a way that, except for references to the basic results in Chapter 1, each is essentially a stand-alone entity that can be read with little reference to other chapters. Similarly, many of the sections in each chapter can be treated in that same independent way.

Chapter 2 starts by presenting the basic setting for Sturm-Liouville boundary value problems. In the second section, sufficient conditions a for the existence of a nontrivial solution to nonlinear Sturm-Liouville systems are presented. Section 3 is concerned with the existence of multiple solutions, while Section 4 presents sufficient conditions for the existence of infinitely many solutions.

In Chapter 3, multipoint problems are discussed. After a brief introduction, the next three sections parallel those in Chapter 2 by examining sufficient conditions for the existence of a nontrivial solution, the existence of multiple solutions, and the existence of infinitely many solutions, respectively. Section 5 deals with two parameter systems and the final section in the chapter contains results on the existence of solutions proved by using the dual action principle.

Chapter 4 is concerned with impulsive boundary values problems, and in addition to discussing the existence of one and infinitely many solutions, the last section in the chapter is devoted to anti-periodic problems.

Partial differential equations is the focus of Chapter 5. Topics include Kirchhoff problems with two parameters, biharmonic systems, elliptic problems with nonstandard growth conditions, and elliptic systems of Kirchhoff type.

Chapter 6 is devoted to difference equations. Periodic problems with one parameter are discussed in Section 2, periodic problems with two parameters in Section 3, multipoint problems with several parameters in Section 4, and homoclinic solutions in Section 5. The final section in the chapter is concerned with anti-periodic solutions of higher order difference equations.

Some brief notes are given in Chapter 7. More than 300 references to the current literature are included.

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John R. Graef
Lingju Kong

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Chapter 1

Mathematical Preliminaries

1.1 Mathematical Preliminaries

In this chapter, we gather together some of the primary mathematical tools that will be used throughout this monograph in proving the existence of solutions to boundary value problems of different types. For the reader's convenience, in the second section in this chapter we present definitions of some of the concepts used in these theorems.

We are interested in results that guarantee the existence of at least one solution, the existence of multiple solutions, and the existence of infinitely many solutions to various types of boundary value problems. Theorems 1.1.1–1.1.16 below will be extensively used in this process. The first two are consequences of an existence result for a local minimum of a functional ([39, Theorem 3.1]) that was inspired by Ricceri's variational principle (see [244]).

To begin, we let X be a nonempty set and let $\Phi,\ \Psi : X \to \mathbb{R}$ be functionals. Define

$$\eta(r_1, r_2) = \inf_{v\in\Phi^{-1}(r_1,r_2)} \frac{\sup_{u\in\Phi^{-1}(r_1,r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)} \tag{1.1.1}$$

and

$$\rho(r_1, r_2) = \sup_{v\in\Phi^{-1}(r_1,r_2)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u)}{\Phi(v) - r_1} \tag{1.1.2}$$

for all $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, and

$$\rho_2(r) = \sup_{v\in\Phi^{-1}(r,\infty)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{\Phi(v) - r}$$

for all $r \in \mathbb{R}$. In what follows, X^* will always denote the dual space of X.

Theorem 1.1.1. ([39, Theorem 5.1]) *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Let $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ such that*

$$\eta(r_1, r_2) < \rho(r_1, r_2).$$

Then, for each $\lambda \in (1/\rho(r_1, r_2), 1/\eta(r_1, r_2))$, there exists $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Another result in this same spirit is the following.

Theorem 1.1.2. ([39, Theorem 5.3]) *Let X be a real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Choose r so that $\inf_X \Phi < r < \sup_X \Phi$, $\rho_2(r) > 0$, and for each $\lambda > \frac{1}{\rho_2(r)}$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive. Then, for each $\lambda \in (\frac{1}{\rho_2(r)}, \infty)$, there exists $u_{0,\lambda} \in \Phi^{-1}(r, \infty)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(r, \infty)$ and $I'_\lambda(u_{0,\lambda}) = 0$.*

These results and their many variations have often been used to obtain multiplicity results for nonlinear problems of a variational nature. See, for example, [23, 24, 39, 40, 244] and the references therein.

To prove the existence of multiple solutions to Sturm-Liouville problems, we will make use of the two following three critical point theorems. The first one was obtained in [22] and is a more precise version of Theorem 3.2 in [24]. It requires the coercivity of the functional $I_\lambda = \Phi - \lambda\Psi$. The second result appeared in [40] and needs a certain sign condition on the functionals.

Theorem 1.1.3. ([40, Theorem 3.2], [42, Theorem 2.2]) *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, and $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact with*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0. \tag{1.1.3}$$

Assume that there is a positive constant r and an element $\overline{v} \in X$, with $2r < \Phi(\overline{v})$, such that:

(a_1) $\frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(a_2) *for all* $\lambda \in \left(\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}\right)$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \left(\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}\right)$, *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points.*

Theorem 1.1.4. ([40, Theorem 3.3], [42, Theorem 2.3]) *Let* X *be a reflexive real Banach space,* $\Phi : X \longrightarrow \mathbb{R}$ *be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on* X^*, *and* $\Psi : X \longrightarrow \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose derivative is compact with:*

(i) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(ii) *for each* $\lambda > 0$ *and for every pair of local minima* u_1, u_2 *of the functional* $\Phi - \lambda\Psi$ *such that* $\Psi(u_1) \geq 0$ *and* $\Psi(u_2) \geq 0$, *we have*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 *and* $\overline{v} \in X$, *with* $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, *such that:*

(b_1) $\frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$;

(b_2) $\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each

$$\lambda \in \left(\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}, \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}\right\}\right),$$

the functional $\Phi - \lambda\Psi$ *has at least three distinct critical points that lie in* $\Phi^{-1}(-\infty, r_2)$.

Since we are also interested in obtaining sufficient conditions for the existence of infinitely many solutions to Sturm-Liouville problems, we will need the following critical points theorem; see [54, Theorem 2.1] and [244, Theorem 2.5].

Theorem 1.1.5. *Let* X *be a reflexive real Banach space and let* Φ, Ψ : $X \to \mathbb{R}$ *be two Gâteaux differentiable functionals such that* Φ *is sequentially* *weakly lower semicontinuous, strongly continuous and coercive, and* Ψ *is sequentially weakly upper semicontinuous. For every* $r > \inf_X \Phi$*, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}, \tag{1.1.4}$$

and

$$\gamma := \liminf_{r\to\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) *For every* $r > \inf_X \Phi$ *and every* $\lambda \in (0, 1/\varphi(r))$*, the restriction of the functional* $I_\lambda := \Phi - \lambda\Psi$ *to* $\Phi^{-1}(-\infty, r)$ *admits a global minimum, which is a critical point (local minimum) of* I_λ *in* X*.*

(b) *If* $\gamma < \infty$*, then for each* $\lambda \in (0, 1/\gamma)$*, the following alternative holds: either*

   (b_1) I_λ *possesses a global minimum, or*

   (b_2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that*

$$\lim_{n\to\infty} \Phi(u_n) = \infty.$$

(c) *If* $\delta < \infty$*, then for each* $\lambda \in (0, 1/\delta)$*, the following alternative holds: either*

   (c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ*, or*

   (c_2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* I_λ *that converges weakly to a global minimum of* Φ*.*

The following three critical points theorem due to Ricceri [248] is useful for showing the existence of three solutions to systems of multi-point boundary value problems.

Theorem 1.1.6. ([248, Theorem 1]) *Let* X *be a real reflexive Banach space,* $I \subseteq R$ *an interval,* $\Phi : X \longrightarrow R$ *a sequentially weakly lower semicontinuous* C^1 *functional that is bounded on each bounded subset of* X *and whose derivative admits a continuous inverse on* X^**, and let* $J : X \longrightarrow R$ *be a* C^1 *functional with a compact derivative. Assume that*

$$\lim_{||x||\to\infty} (\Phi(x) + \lambda J(x)) = \infty \tag{1.1.5}$$

for all $\lambda \in I$, *and that there exists* $\rho \in R$ *such that*

$$\sup_{\lambda\in I}\inf_{x\in X}(\Phi(x)+\lambda(J(x)+\rho)) < \inf_{x\in X}\sup_{\lambda\in I}(\Phi(x)+\lambda(J(x)+\rho)). \quad (1.1.6)$$

Then, there exist a non-empty open interval $A \subseteq I$ *and a positive real number* q *with the following property: for every* $\lambda \in A$ *and every* C^1 *functional* $\Psi : X \longrightarrow R$ *with compact derivative, there exists* $\delta > 0$ *such that, for each* $\mu \in [0, \delta]$, *the equation*

$$\Phi'(x) + \lambda J'(x) + \mu\Psi'(x) = 0$$

has at least three solutions in X *whose norms are less than* q.

In order to apply Theorem 1.1.6 above, the following result, which is a restatement of Proposition 1.3 in [39] (J is replaced by $-J$), is useful in showing that inequality (1.1.6) holds.

Theorem 1.1.7. ([39, Proposition 1.3]) *Let* X *be a non-empty set and* Φ *and* J *be two real valued functionals defined on* X. *Assume that* $\Phi(u) \geq 0$ *for every* $u \in X$ *and there exists* $u_0 \in X$ *such that* $\Phi(u_0) = J(u_0) = 0$. *Furthermore, assume that there exist* $w \in X$ *and* $r > 0$ *such that* $\Phi(w) > r$ *and*

$$\sup_{\Phi(u)<r}(-J(u)) < r\frac{-J(w)}{\Phi(w)}.$$

Then, for every $h > 1$ *and for every* $\rho \in R$ *satisfying*

$$\sup_{\Phi(u)<r}(-J(u)) + \frac{r\frac{-J(w)}{\Phi(w)} - \sup_{\Phi(u)<r}(-J(u))}{h} < \rho < r\frac{-J(w)}{\Phi(w)},$$

we have

$$\sup_{\lambda\in R}\inf_{u\in X}(\Phi(u)+\lambda(J(u)+\rho)) < \inf_{u\in X}\sup_{\lambda\in[0,\upsilon]}(\Phi(u)+\lambda(J(u)+\rho)),$$

where

$$\upsilon = \frac{hr}{r\frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(u)<r}(-J(u))}.$$

We next define what is meant by the Palais–Smale condition.

Definition 1.1.1. *Let* X *be a real Banach space. The functional* $J \in C^1(X, \mathbb{R})$ *is said to satisfy the Palais–Smale (PS) condition if every sequence* $\{u_n\} \subset X$, *such that* $J(u_n)$ *is bounded and* $J'(u_n) \to 0$ *as* $n \to \infty$, *has a convergent subsequence. Here, the sequence* $\{u_n\}$ *is called a (PS) sequence.*

The following result is the well-known Mountain Pass Theorem of Ambrosetti and Rabinowitz [241] (also see [185, Theorem 7.1]). Here, $B_r(u)$ is the open ball centered at $u \in X$ with radius $r > 0$, $\overline{B}_r(u)$ is its closure, and $\partial B_r(u)$ is its boundary.

Theorem 1.1.8. *Let $(X, \|\cdot\|)$ be a real Banach space and $I \in C^1(X, \mathbb{R})$. Assume that I satisfies the (PS) condition and there exist $u_0, u_1 \in X$ and $\rho > 0$ such that:*

(A1) $u_1 \notin \overline{B}_\rho(u_0)$;
(A2) $\max\{I(u_0), I(u_1)\} < \inf_{u \in \partial B_\rho(u_0)} I(u)$.

Then, I possesses a critical value which can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)) \geq \inf_{u \in \partial B_\rho(u_0)} I(u),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) \ : \ \gamma(0) = u_0, \ \gamma(1) = u_1\}.$$

In the following definition, we present an important notion known as the Ambrosetti–Rabinowitz condition.

Definition 1.1.2. *A function $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is said to satisfy an Ambrosetti–Rabinowitz type condition if*

(AR) *there exists $\mu > p$ and $R > 0$ such that*

$$0 \leq \mu F(k,t) \leq t f(k,t) \quad \text{for } (k,t) \in \mathbb{Z} \times \mathbb{R} \text{ with } |t| \geq R, \quad (1.1.7)$$

where $F(k,t) = \int_0^t f(k,s)ds$.

Condition (AR) is often used to show that any (PS) sequence of the corresponding energy functional is bounded. This plays an important role in the application of critical point theory.

The following three theorems will be useful in proving existence results for boundary value problems for difference equations. Theorem 1.1.9 below can be found in [227, 289], Theorem 1.1.10 in [66], and Theorem 1.1.11 in [89, 241].

Theorem 1.1.9. *Let X be a real reflexive Banach space, and let J be a weakly upper (lower, respectively) semicontinuous functional such that*

$$\lim_{||u|| \to \infty} J(u) = -\infty \quad \left(\lim_{||u|| \to \infty} J(u) = \infty, \ \textit{respectively} \right).$$

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