

SECOND EDITION

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# Foundations of Mechanics

Revised, enlarged, reset

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Ralph Abraham  
Jerrold E. Marsden

# Foundations of Mechanics

Second Edition

Revised, enlarged, and reset

*A mathematical exposition of classical mechanics  
with an introduction to  
the qualitative theory of dynamical systems  
and applications to the three-body problem*

**RALPH ABRAHAM AND JERROLD E. MARSDEN**

*University of California*

*Santa Cruz and Berkeley*

with the assistance of

**Tudor Ratiu and Richard Cushman**



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## Preface to the Second Edition

Since the first edition of this book appeared in 1967, there has been a great deal of activity in the field of symplectic geometry and Hamiltonian systems. In addition to the recent textbooks of Arnold, Arnold-Avez, Godbillon, Guillemin-Sternberg, Siegel-Moser, and Souriau, there have been many research articles published. Two good collections are "Symposia Mathematica," vol. XIV, and "Géométrie Symplectique et Physique Mathématique," CNRS, Colloque Internationaux, no. 237. There are also important survey articles, such as Weinstein [1977b]. The text and bibliography contain many of the important new references we are aware of. We have continued to find the classic works, especially Whittaker [1959], invaluable.

The basic audience for the book remains the same: mathematicians, physicists, and engineers interested in geometrical methods in mechanics, assuming a background in calculus, linear algebra, some classical analysis, and point set topology. We include most of the basic results in manifold theory, as well as some key facts from point set topology and Lie group theory. Other things used without proof are clearly noted.

We have updated the material on symmetry groups and qualitative theory, added new sections on the rigid body, topology and mechanics, and quantization, and other topics, and have made numerous corrections and additions. In fact, some of the results in this edition are new.

We have made two major changes in notation: we now use  $f^*$  for pull-back (the first edition used  $f_*$ ), in accordance with standard usage, and have adopted the "Bourbaki" convention for wedge product. The latter eliminates many annoying factors of 2.

A. N. Kolmogorov's address at the 1954 International Congress of Mathematicians marked an important historical point in the development of the theory, and is reproduced as an appendix. The work of Kolmogorov, Arnold, and Moser and its application to Laplace's question of stability of the solar system remains one of the goals of the exposition. For complete details of all the theorems needed in this direction, outside references will have to be consulted, such as Siegel-Moser [1971] and Moser [1973a].

We are pleased to acknowledge valuable assistance from

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Richard Cushman and Tudor Ratiu

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Further, we express our gratitude to Chris Shaw, who made exceptional efforts to transform our sketches into the graphics which illustrate the text, to Peter Cocha for his assistance in organizing the Museum and Bibliography, and to Ruthie Cephas, Jody Hilbun, Marnie McElhiney, Ruth (Bionic Fingers) Suzuki, and Ikuko Workman for their superb typing job.

Theoretical mechanics is an ever-expanding subject. We will appreciate comments from readers regarding new results and shortcomings in this edition.

RALPH ABRAHAM  
JERROLD E. MARSDEN

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## Preface to the First Edition

In the Spring of 1966, I gave a series of lectures in the Princeton University Department of Physics, aimed at recent mathematical results in mechanics, especially the work of Kolmogorov, Arnold, and Moser and its application to Laplace's question of the stability of the solar system. Mr. Marsden's notes of the lectures, with some revision and expansion by both of us, became this book.

Although the lectures were attended equally by mathematicians and physicists, our goal was to make the subject available to the nonspecialists. Therefore, the mathematical background assumed was dictated by the physics graduate students in the audience. Hoping this would be typical of the people interested in this subject, I have made the same assumptions in the book.

Thus, we take for granted basic undergraduate calculus and linear algebra, and a limited amount of classical analysis, point set topology, and elementary mechanics. Then we begin with modern advanced calculus, and go on to a complete and self-contained treatment of graduate level classical mechanics. The later chapters, dealing with the recent results, require an ever-increasing adeptness in general topology, and we have collected the topological topics required in Appendix A.

To further aid the nonmathematician, the proofs are unusually detailed, and the text is replete with cross-references to earlier definitions and propositions, all of which are numbered for this purpose. The extent of these is testimony of Mr. Marsden's patience.

As our goal is to make a concise exposition, we prove propositions only if the proofs are easy, or are not to be found readily in the literature. This

results in an irregular collection of proofs—in the first four chapters nearly everything is proved, being easy, and in the last three chapters there are several longer proofs included and many omitted. Some of those included are necessary because the propositions are original, and can be omitted in a first reading or an elementary course.

For the mathematical reader, the proofs we have omitted can easily be found in books or journals, and we give complete references for each (References in square brackets refer to the Bibliography.) For this reason, the book, although not self-contained, gives a complete exposition.

In this connection we are grateful to Al Kelley for the opportunity of publishing two research articles of his, as Appendixes B and C, which have not appeared elsewhere. In each of these he proves an original theorem that is important to our development of the subject. As Kolmogorov's address at the 1954 International Congress of Mathematicians (in Russian), which inspired the most important of the recent results, has not been available in English, we include a translation of it in Appendix D. The exercises at the end of each section are nearly all used in a later section, and may be read as part of the text.

I am indebted to Arthur Wightman for his enthusiasm in making arrangements for my lectures and the publication of the book, to René Thom for discussions on structural stability and a preliminary manuscript of part of his book on that subject, to Jerrold Marsden for his energetic collaboration in the writing of this book, and to many colleagues for valuable suggestions. Some of these are acknowledged in the Notes at the end of each chapter, which also give general historical and bibliographical information.

We are both happy to express our gratitude to June Clausen for editing and typing the bulk of the manuscript, and to Patricia Clark, Bonnie Kearns, Elizabeth Epstein, Elizabeth Margosches, and Jerilynn Christiansen for their valuable assistance.

RALPH ABRAHAM

Princeton, New Jersey  
June 1966

**Museum**



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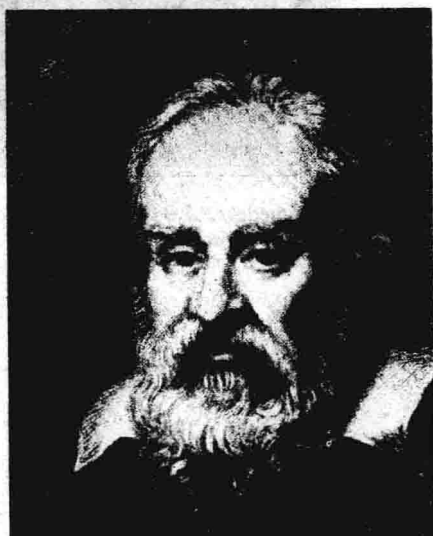
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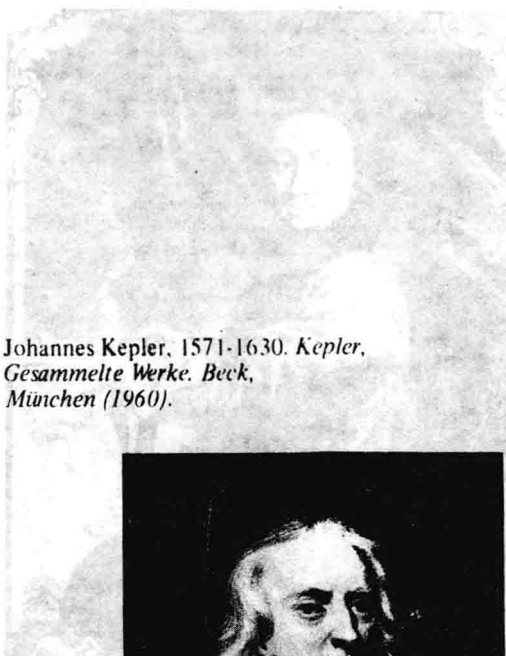
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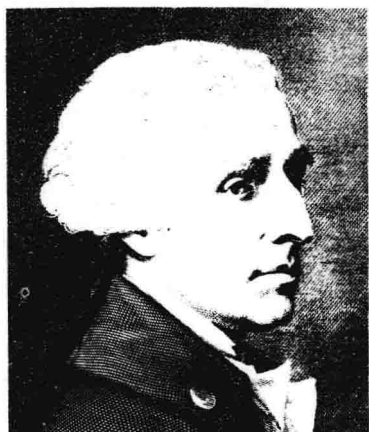
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