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Jan de Vries

TOPOLOGICAL DYNAMICAL SYSTEMS

AN INTRODUCTION TO THE DYNAMICS OF CONTINUOUS MAPPINGS

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An Introduction to the Dynamics of Continuous Mappings

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Topological Dynamical Systems

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Preface

Thus the superior man understands the transitory
in the light of the eternity of the end.
After: I Ching, Hexagram 54.

This book is just what its title says: an introduction. It is addressed primarily to graduate students who want to learn the basic ideas of topological dynamics. Thus, the fundamental notions of (topological) dynamical systems are defined and their elementary properties are discussed. Students who have mastered this book will have a firm basis to start research related with the topics discussed here, even though we have not included all most recent results. Unfortunately, in order to keep this book reasonably sized many important topics could not be included (or even mentioned). The choice of which topics to include is to a great extend determined by my wish to concentrate on the purely topological aspects of the theory, in the spirit of the work by Birkhoff.

When I started writing [deV][1], '(topological) dynamics' was a relatively unknown topic. At present the situation is completely different and one might say that this topic is quite popular, to the extend that introductory courses in differential equations are sometimes called courses in dynamical systems. On the other hand, the purely topological approach to dynamical systems theory gets less attention than it deserves. Therefore I decided to publish the lecture notes of a course in 'applied topology' that I gave at the Free University in Amsterdam from 1995 until my retirement in 2002.

The process of transforming my Dutch lecture notes into an English book was more complicated and took much more time than anticipated. Not only had the material to be reorganized, streamlined and expanded, but also a severe illness prevented me from working during a couple of years.

The book you have before you treats systems consisting of a topological space (the phase space) and iterations of a single continuous mapping of this space into itself (the phase mapping). Except in Section 2.1 no use is made of differentiability of the phase mapping. The assumption that the phase space is metrizable is avoided as much as possible.

The book is organized as follows: In the Introduction the 'dynamical systems approach' is explained: the philosophy behind the material in the book and the red thread through the subsequent chapters. Though most of the Introduction contains no specific results needed in the rest of the book, the reader is strongly advised to have a look at it. Moreover, the examples at the end of the Introduction are heavily used later on (but reading them can be postponed until they are needed).

The Chapters 1 and 2 together with the examples in the Introduction are used throughout the remainder of the book. In Chapter 1 the basic notions of the theory are

1 Items like [deV] refer to the literature.

defined, the elementary properties of dynamical systems are discussed and illustrated by examples. Chapter 2 treats dynamical systems on intervals in $\mathbb{R}$; it culminates in a proof of Šarkovskij's Theorem.

The Chapters 3 and 4 are about stability: stability of invariant sets and variants of Poisson-stability (which we call 'recurrence'), including almost periodic and non-wandering points. Chapter 3 also discusses attraction, though we refrain from giving a formal definition of an 'attractor'. See, however, the Notes at the end of that chapter. Chapter 3 ends with a discussion of the space of components of a transitive (asymptotically) stable set in a locally connected locally compact phase space – providing full proofs of statements that have unconvincing proofs elsewhere. The discussion of recurrence and almost periodicity in Chapter 4 is restricted to a bare minimum because there is much other literature about these topics: see [GH] and [deV] to get an idea.

Next, in Chapter 5 we discuss shift systems (spaces of sequences with the shift operator) and in Chapter 6 we investigate how such systems can be used to represent other systems by means of a suitable coding (symbolic dynamics). The study of shift systems has a strong algebraic/combinatorial flavour and it has many applications in and points of contact with other parts of mathematics, from artificial languages to coding theory and from automata theory to probability theory. We discuss none of these applications; the interested reader is referred to D. Lind & B. Marcus [1995] or B. Kitchens [1998]. We give only some applications of symbolic dynamics to 1-dimensional systems; these are used in Chapter 8 to compute the topological entropy of those systems.

Chapter 7 deals with notions of chaos. There are many definitions of this notion, all about 'erratic behaviour'. We concentrate on sensitive dependence on initial conditions and the existence of large so-called 'scrambled' sets. In Chapter 8 we discuss the notion of topological entropy. We include a proof of the fact that, for maps of an interval into itself, positive entropy is equivalent to the existence of a point with odd primitive period greater than 1 which, by the results of Chapter 7, implies chaos.

The results of the Chapters 3 and 4 are not needed for a good understanding of the later chapters, though in Chapter 5 some examples are given that illustrate notions and results from these previous chapters. Similarly, the Chapters 5 and 6 are not needed for an understanding of the Chapters 7 and 8, though also here examples in the latter chapters are taken from the former. So possible courses can be be based on the Chapters 1 and 2 followed by either the Chapters 3 and/or 4, or 5 and 6, or 7 and 8. Of course, other selections are possible, depending on the available time.

Every chapter concludes with a set of exercises. Most of them are routine applications of the material in the chapter, others deal with extensions of the theory. For the more challenging ones one may find hints (or if you prefer: telegramm-style answers) at the end of the book. The exercises are followed by a section of Notes in which also references to the literature are given. These Notes are rather sketchy and the references are far from complete. In particular, they are not meant as complete historical introductions; rather, they tell how the results came to me. In point of fact, many results included in the book are common knowledge and are known already for decennia. My knowledge of dynamical systems grew over a rather long period and often I lost track of where I read or heard the various results. Consequently, for many results in this book references to the original sources are missing. But I can safely state that I learned the 'classical' results from the book [GH]. I started reading [GH] around 1970 (I was interested in almost periodic functions and transformation groups) but after a couple of hours I threw the book aside in despair (some outstanding mathematicians told me they had the same experience) – try it, and you will understand why. A couple of months later I tried again, then a year later.... Over the years I learned how to extract information from that book and about 1985 I could reasonably find my way in it. It really contains almost everything that was know about topological dynamics in, say, 1950. Of course, much in the present book was discovered later, and I have tried to give credit to whom it deserves.

Finally, a few words about terminology. In the literature the objects studied in this book are often called *semi*-dynamical systems. Moreover, what we call orbits, or limit sets, or invariant sets, are usually called *positive semi-orbits*, *positive (or omega-)limit sets*, *positively (or forward) invariant sets*, etc. As we pay no special attention to invertible systems and, consequently, no negative semi-orbits, or negative (or alpha-)limit sets, or backward invariant sets, etc., are discussed, it would seem somewhat redundant to prefix all notions with 'semi-', 'positive(ly)' or 'forward'. So I decided to omit all those prefixes. This has the disadvantage that the terminology in this book does not always agree with the usual one in the literature. I have tried to obviate difficulties caused by conflicting terminology by adding remarks on 'invertible vs. non-invertible systems' at the end of the relevant chapters. These notes do not cover all differences: I tried to restrict myself to the discrepancies which I believe are likely to cause confusion. The reader who gets confused by seemingly conflicting results in this book on the one hand and the literature on the other is advised to have a look at these notes.

The prerequisites for understanding this book are are rather modest. A reader who has mastered a course in General Topology and has a working knowledge of Calculus should be able to follow all arguments. For a good understanding of the Introduction some familiarity with the theory of differential equations is useful. For easy reference there are two Appendices at the end of the book: one with the preliminaries from general topology and a second one about the Cantor set.

The internal reference system is rather straightforward. The numbering of sections starts anew in each chapter; thus, 'Section $k.m$' refers to the m-th section in Chapter k. Within each section the items are numbered consecutively: '$k.m.n$' refers to item n in Section $k.m$. Equations are numbered separately with the a similar system, but a slightly different style is used: '($k.m$-n)' refers to the n-th formula in Section $k.m$. Figures and Exercises are numbered as follows: 'Figure $k.n$' refers to the n-th figure in Chapter k and 'Exercise $k.n$' refers to the n-th exercise at the end of Chapter k; moreover, 'Exercise $k.n$ (i)' refers to part (i) of Exercise $k.n$.

Finally, I want to express my indebtedness to the people who contributed to my (mathematical) education and my knowledge of dynamical systems. First, I mention the late prof. J. F. Koksma. With a course on 'Almost Periodic Functions' (covering most of Maak's book) he roused my interest in almost periodicity. Most of my later research was in some sense related to this topic. Next, prof. P. C. Baayen (Cor) played a decisive role in my mathematical life. He was my Ph. D. thesis supervisor and, as head of the Pure Mathematics Department of the 'Mathematisch Centrum' in Amsterdam – presently called CWI – he invited me to work there after my graduation. Later, as Scientific Director of the CWI (1980–1994) he directly contributed to the ideal research environment that the CWI was at that time, and in this way he indirectly contributed to my research. His broad knowledge of mathematics (not to mention history, Tolkien, science fiction, ...) has always inspired me. In the 1980's I supervised one of his Ph. D. students, Jaap van der Woude, in his research in topological dynamics – or rather, Jaap took me in tow and I had to work quite hard to keep pace with him. This cooperation eventually gave rise to the book [deV]. I also want to mention Jan Aarts; my lectures at the Free University were based on the lecture notes of a course he gave at the Technical University in Delft. Finally, I owe much to Mike Keane and the guest speakers at his seminars in Delft and in Amsterdam.

My last words here are for my wife Liet. This month we celebrated the fact that we first met exactly 50 years ago. Since, her caring love has sustained me. It kept me, literally, in life during the preparation of this book.

Opende, november 2013 *Jan de Vries*

Notation

The best notation is no notation...
Paul Halmos, How to write Mathematics (1970)

The purpose of the list of symbols below is twofold. First we establish some general notation concerning sets and mappings. With a minor exception, this notation is not defined in this book and it will be used without further reference. Also some notation about topological and metric spaces is collected. Most of it is defined in Appendix A, but the descriptions below should be sufficiently clear and no references to particular page numbers are given. Secondly, in this list we collect the notation about dynamical systems developed in this book. Here the page numbers where the respective notions are defined are added between square brackets.

Sets and mappings

- $\left.\begin{array}{r} P := Q \\ Q =: P \end{array}\right\}$: P is defined as Q.
- $A \subseteq B$: A is a subset of B (possibly $A = B$).
- $A \subset B$: $A \subseteq B$ and $A \neq B$.
- $\mathbb{R}$: the real line.
- $\mathbb{Z}$: the integers.
- $\mathbb{Q}$: the set of rational numbers.
- $\mathbb{R}^+ := \{s \in \mathbb{R} : s \geq 0\}$.
- $\mathbb{Z}^+ := \mathbb{Z} \cap \mathbb{R}^+$.
- $\mathbb{N} := \mathbb{Z}^+ \setminus \{0\}$.
- $\mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}$ (the circle).
- $sA + tB := \{sa + tb : a \in A \text{ and } b \in B\}$ for $A, B \subseteq \mathbb{R}$ and $s, t \in \mathbb{R}$.
- $[a; b] := \{s \in \mathbb{R} : a \leq s \leq b\}$ (closed interval).
- $(a; b) := \{s \in \mathbb{R} : a < s < b\}$ (open interval)
- $(a; b] := \{s \in \mathbb{R} : a < s \leq b\}$ (left open, right closed interval).
- $[a; b) := \{s \in \mathbb{R} : a \leq s < b\}$ (left closed, right open interval).

If J is a bounded interval then $|J|$ will denote the length of J. So if J is one of the above intervals then $|J| = b - a$.

- $[t] := e^{2\pi i t} \in \mathbb{S}$ for $t \in \mathbb{R}$.
- $d_c([s], [t]) := 2\pi \min\{|s - t| (\mathrm{mod}\ 1), 1 - |s - t| (\mathrm{mod}\ 1)\}$ for $s, t \in \mathbb{R}$ (metric in $\mathbb{S}$).

If X is a set then $\mathrm{id}_X : x \mapsto x : X \to X$ is the identity mapping. If $n \in \mathbb{N}$ then $X^n := X \times \cdots \times X$ (n times) and $\Delta_X := \{(x, \ldots, x) \in X^n : x \in X\}$ (used almost exclusively in the case that $n = 2$).

Let $f: X \to Y$ be a mapping, $A \subseteq X$, $B \subseteq Y$ and $y \in Y$. Then:
- $f[A] := \{f(x) : x \in A\}$.
- $f^{\leftarrow}[B] := \{x \in X : f(x) \in B\}$.
- $f^{\leftarrow}[y] := f^{\leftarrow}[\{y\}]$.

If f is a bijection then the *inverse* of f is denoted f^{-1}. In that case:
- $f^{\leftarrow}[B] = f^{-1}[B]$ and $f^{\leftarrow}[y] = \{f^{-1}(y)\}$.

If $I \subseteq X$ and $J \subseteq Y$ then:
- $f : I \overset{\circ}{\to} J$ means: $f[I] \supseteq J$ (f maps I over J) [80],
- $f : I \twoheadrightarrow J$ means: $f[I] = J$ (f maps I onto J) [80].

Topological and metric spaces

Let X be a topological space, $x \in X$ and $A \subseteq X$. Then:
- $\mathcal{N}_x$: the set of all neighbourhoods of the point x.
- $\mathcal{N}_A$: the set of all neighbourhoods of the set A.
- $\overline{A}$, A^-, $\mathrm{cl}_X(A)$ (or $\mathrm{cl}(A)$ if X is understood) the closure of A in X.
- A°, $\mathrm{int}_X(A)$ (or $\mathrm{int}(A)$ if X is understood) the interior of A.
- $C(X, X)$ the set of all continuous mappings from X into itself.

Let (X, d) be a metric space, $x \in X$, $A \subseteq X$ and $\varepsilon > 0$. Then:
- $B_\varepsilon(x) := \{x' \in X : d(x, x') < \varepsilon\}$ (open ball about x with radius ε).
- $S_\varepsilon(x) := \{x' \in X : d(x, x') \leq \varepsilon\}$ (closed ball about x with radius ε).
- $d(x, A) := \inf\{d(x, y) : y \in A\}$ (distance between x and A).
- $B_\varepsilon(A) := \{x' \in X : d(x', A) < \varepsilon\} = \bigcup\{B_\varepsilon(y) : y \in A\}$ (open ε-neighbourhood of A).
- $C_c(X, Y) : C(X, Y)$ endowed with the compact-open topology.
- $C_u(X, Y) : C(X, Y)$ endowed with the topology of uniform convergence (Y a metric space).

Continuous functions defining dynamical systems

- $f_\mu : x \mapsto \mu x(1 - x) : \mathbb{R} \to \mathbb{R}$ for $\mu > 0$: the quadratic or logistic family [9].
- $\varphi_a : [t] \mapsto [a + t] : \mathbb{S} \to \mathbb{S}$ for $a \in \mathbb{R}$: the rigid rotation of the circle [11].
- $\psi : [t] \mapsto [2t] : \mathbb{S} \to \mathbb{S}$: the argument-doubling transformation [12].
- $T : x \mapsto \min\{2x, 2(1 - x)\} : [0; 1] \to [0; 1]$: the tent map [9].
- $T_\lambda : x \mapsto \min\{T(x), \lambda\} : [0; 1] \to [0; 1]$: the truncated tent map [84].
- $T_s : x \mapsto \frac{s}{2}(1 - |2x - 1|) : [0; 1] \to [0; 1]$: the generalized tent map [314].
- $\sigma : (x_n)_{n \in \mathbb{Z}^+} \mapsto (x_{n+1})_{n \in \mathbb{Z}^+} : \Omega_S \to \Omega_S$: the shift (shift map) [223].
- σ_X: the shift map restricted to a subshift X [226].

Dynamical notions

Let (X, f) be a dynamical system, $x \in X$ and $A, B \subseteq X$. Then:
- $\mathcal{O}_f(x) := \{f^n(x) \,\vdots\, n \in \mathbb{Z}^+\}$: the orbit of x under f [7,17].
- $\omega_f(x) := \bigcap_{n=0}^{\infty} \overline{\mathcal{O}_f(f^n(x))}$: the (positive) limit set of x [33].
- $D(x, B) := \{n \in \mathbb{Z}^+ \,\vdots\, f^n(x) \in B\}$: the dwelling set of x in B [28].
- $D(A, B) := \{n \in \mathbb{Z}^+ \,\vdots\, f^n[A] \cap B \neq \emptyset\} = \{n \in \mathbb{Z}^+ \,\vdots\, A \cap (f^n)^{\leftarrow}[B] \neq \emptyset\}$: the dwelling set of A in B [31].
- $\mathcal{B}_f(A) := \{x \in X \,\vdots\, \emptyset \neq \omega_f(x) \subseteq A\}$: the basin of attraction of A (A not empty, closed and invariant) [120].
- $R(X, f)$: the set of recurrent points of (X, f) [165].
- $Z(X, f) := \overline{R(X, f)}$: the centre of (X, f) [178].
- $\Omega(X, f)$: the non-wandering set of (X, f) [175].
- $CR(X, f)$: the chain recurrent set of (X, f) [185].
- $\mathbf{Trans}(X, f)$: the set of all transitive points in (X, f) [66].
- $\mathbf{Eq}(X, f)$: the set of all points x such that the system is equicontinuous (that is, stable) at x [326].

If the phase space X is a metric space:
- $d_n^f(x, y) := \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y))$ $(x, y \in X, n \in \mathbb{N})$ [380].
- $h(K, f)$: the topological entropy of K under f ($K \subseteq X$ compact) [382].
- $h(f)$: the topological entropy of f [382].

If the phase space X is a any space and $\mathcal{A}$ and $\mathcal{B}$ are covers of X then:
- $\mathcal{A} < \mathcal{B}$: $\mathcal{B}$ is *finer than* $\mathcal{A}$ (or: $\mathcal{A}$ is *coarser than* $\mathcal{B}$) [395].
- $\mathcal{A} \vee \mathcal{B}$: the *join* of the covers $\mathcal{A}$ and $\mathcal{B}$ [395].
- $\mathcal{A}^{f,n} := \mathcal{A} \vee f^{\leftarrow}\mathcal{A} \vee \cdots \vee (f^{n-1})^{\leftarrow}\mathcal{A}$ ($\mathcal{A}$ a cover, $n \in \mathbb{N}$) [395]
- $N(\mathcal{A})$: the minimal cardinality of a finite subcover of a special cover $\mathcal{A}$ [396].
- $H(\mathcal{A}) := \log N(\mathcal{A})$ [396].
- $h(f, \mathcal{A})$: the entropy of f with respect to the special cover $\mathcal{A}$ [397].

Shift spaces and symbolic representations

Let $\mathbf{S}$ be a finite set with at least two elements.
- $\mathbf{S}^* := \bigcup_{n=0}^{\infty} \mathbf{S}^n$: the langue – or alphabet – over the symbol set $\mathbf{S}$ [219].
- $\square$: the unique element of $\mathbf{S}^0$, i.e., the empty word [219].
- $\Omega_{\mathbf{S}} := \mathbf{S}^{\mathbb{Z}^+}$: the (full) shift space over the symbol set $\mathbf{S}$ [218].
- $\widetilde{B}_k(x) := \{y \in \Omega_{\mathbf{S}} \,\vdots\, y_i = x_i \text{ for } 0 \leq i \leq k-1\}$: the cylinder about the point $x \in \Omega_{\mathbf{S}}$, based on the initial block of x with length $k \geq 1$ [221].
- $\mathfrak{X}(\mathcal{B}) := \{x \in \Omega_{\mathbf{S}} \,\vdots\, \text{no member of } \mathcal{B} \text{ occurs in } x\}$ ($\mathcal{B} \subseteq \mathbf{S}^* \setminus \{\square\}$) [228].

- $\mathcal{A}(X) := \{b \in \mathbf{S}^* : b$ does not occur in any point of $X\}$ (X a shift space): the words absent from X [227].
- $\mathcal{L}(X) := \mathbf{S}^* \setminus \mathcal{V}(X)$ (X a shift space): the language of a shift space X, that is, the set of X-present words [230].
- $\mathcal{A}_k(X) := \{b \in \mathbf{S}^k : b$ does not occur in any point of $X\}$ (X a shift space, $k \in \mathbb{N}$): the words of length k absent from X [234].
- $\mathcal{L}_k(X) := \mathbf{S}^k \setminus \mathcal{V}_k(X)$ (X a shift space, $k \in \mathbb{N}$): the X-present words of length k [234].
- $\mathfrak{M}_v(G)$: the SFT defined by the faithfully vertex-labelled graph G [245].
- $\mathfrak{W}_v(G)$: the shift space defined by the infinite walks on a vertex-labelled graph G [249].
- $\mathfrak{W}_e(G)$: the sofic shift space defined by the infinite walks on an edge-labelled graph G [250].

If $\mathcal{P} := \{P_0, \dots, P_{s-1}\}$ is a topological partition of X and $\mathbf{S} := \{0, \dots, s-1\}$ then
- $\iota(x) \in \Omega_{\mathbf{S}}$, the point with coordinates $\iota(x)_n$ such that $f^n(x) \in \mathcal{P}_{\iota(x)_n}$ for $n \in \mathbb{Z}^+$: the itinerary of a point x with respect to the topological partition $\mathcal{P}$ [283].
- $X^* := X^*(\mathcal{P}, f)$: the set of points $x \in X$ with a full itinerary $\iota(x)$ [283].
- $Z := Z(\mathcal{P}, f)$: the symbolic model of X generated by $\mathcal{P}$ (provided $\mathcal{P}$ is f-adapted) [284/285].
- $D_k(b) := \bigcap_{n=0}^{k-1} (f^n)^{\leftarrow}[P_{b_n}]$: ($\{P_0, \dots, P_{s-1}\}$ an indexed family of subsets of X, $b = b_0 \cdots b_{k-1}$ a k-tuple of symbols from the set $\{0, \dots, s-1\}$) [284, also 407].
- $\psi := \psi_{\mathcal{P},f} : (Z, \sigma_Z) \to (X, f)$: the symbolic representation of (X, f) (provided $\mathcal{P}$ is a pseudo-Markov partition) [287].

Contents

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Unknown author.