An Introduction to Noncommutative Noetherian Rings

Second Edition

K. R. Goodearl and R. B. Warfield, Jr.

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Introduction to the Second Edition

Dedicated to my friend and coauthor,

Robert Breckenridge Warfield, Jr. (1940–1989)

Since the publication of the first edition in 1989, this book has been used by several generations of graduate students. From the accumulated comments, it became clear that a number of changes in the presentation of the material would make the book more accessible, particularly to students reading the text on their own. During this same period, the explosive growth of the area of quantum groups provided a large new crop of noetherian rings to be analyzed, and thus gave major impetus to research in noetherian ring theory. While a general development of the theory of quantum groups would not fit into a book of the present scope, many of the basic types of quantum groups are ideally suited as examples on which the concepts and tools developed in the text can be tested. Finally, readers of the first edition found a substantial list of typographical and other minor errors. This revised edition is designed to address all these points. Undoubtedly, however, the retyping of the text in TeX has introduced a new supply of typos for readers' entertainment.

Here is more detail:

Changes to the order and emphasis of topics were based, as mentioned, on the combined experience and comments of numerous students and professors who used the first edition over the past 14 years. In particular, more examples and additional manipulations with specific rings – especially in the early part of the book – were requested. In response to both requests, further examples of the types discussed in the first edition are worked out, and new examples from quantum groups (as well as a few from the representation theory of Lie algebras) have been inserted throughout. The discussion of skew polynomial rings, which many students initially found difficult to digest in full generality, has been expanded considerably. The present development keeps to the case of twists by automorphisms in Chapter 1 and begins Chapter 2 by outlining the case of twists by derivations; thus, readers have a chance to familiarize themselves with these more basic types of skew polynomial rings before moving on to the general situation. In addition, the universal properties of these rings are now emphasized, as are presentations by generators and relations.

It was also brought to my attention that most students have a strong preference for following the given sequence of topics in a text, as opposed to skipping certain sections and returning to them later. Thus, some topics which were presented in the early chapters of the first edition, but were not essential until later, have been moved. For example, the development of affiliated primes and affiliated series, which originally occurred in the introductory chapter on prime ideals, has now been shifted to Chapter 8. To address a few points in earlier chapters where affiliated series had provided motivation, the (simpler) concept of a prime series is introduced in Chapter 3.

When the first edition was being written, students' opinions on two possible approaches to Goldie's Theorem (the classical construction of rings of fractions versus ring structures on injective hulls) were evenly split among those polled. The injective hull approach was chosen mainly for the sake of variety (to contrast with presentations in other sources). In the meantime, however, opinion has swung overwhelmingly in favor of the classical approach. Consequently, the development of ring structures on injective hulls has been removed. The accompanying material on nonsingular modules has been replaced by a discussion of torsionfree modules with respect to Ore sets. To accommodate the classical approach, a basic construction of rings of fractions (with respect to Ore sets of regular elements) is now given at the beginning of Chapter 6; the general case (in Chapter 10) is tackled by reduction to this basic case. In keeping with one of the main themes of the book, rings of fractions are obtained as rings of endomorphisms of appropriate modules, thus avoiding tedious computations with equivalence classes of ordered pairs.

The topic of quantum groups is a tricky one for an introductory book. Certainly, the algebraic side of that area has provided fertile ground for applications of noetherian ring theory. However, on one hand, the subject – like those of group algebras and enveloping algebras – has given rise to such an extensive theory of its own that a general treatment would completely overbalance the present book. On the other hand, the theory of quantum groups is still evolving rapidly even though its foundations are not yet settled; in fact, there is still no axiomatic definition of a "quantum group" at present, only a list of examples which have been so baptized by general consensus. For these reasons, it did not appear useful, at this point, to attempt an introductory account of the topic trimmed to the length of a chapter or two. Thus, in place of a systematic treatment, quantum groups have been integrated into the general flow of the book to illustrate the theory. Moreover, a sketch of the philosophy behind the concept of a quantum group has been added to the Prologue, to accompany the previous sketches of other areas of application of noetherian ring theory. A selection of easily accessible examples, constructible from iterated skew polynomial rings, is introduced at that point. These examples are analyzed in detail in the first two chapters (in both text and exercises) and are used repeatedly in later chapters to test new concepts and methods.

For many helpful comments and suggestions, most of which I have tried to

incorporate into this revised edition, I would like to thank Allen Bell, Gary Brookfield, Ken Brown, R. N. Gupta, Charu Hajarnavis, Heidi Haynal, Karen Horton, Brian Jue, Dennis Keeler, Tom Lenagan, Ed Letzter, Ian Musson, Kim Retert, Dan Rogalski, Lance Small, Paul Smith, Toby Stafford, Peter Thompson, and Scot Woodward.

Ken Goodearl
July 2003

Introduction to the First Edition

Noncommutative noetherian rings are presently the subject of very active research. Recently the theory has attracted particular interest due to its applications in related areas, especially the representation theories of groups and Lie algebras. We find the subject of noetherian rings an exciting one, for its own sake as well as for its applications, and our primary purpose in writing this volume was to attract more participants into the area.

This book is an introduction to the subject intended for anyone who is potentially interested, but primarily for students who are at the level which in the United States corresponds to having completed one year of graduate study. Since the topics included in an American first year graduate course vary considerably, and since those in analogous courses in other countries (e.g., third year undergraduate or M.Sc. courses in Britain) vary even more, we have attempted to minimize the actual prerequisites in terms of material, by reviewing some topics that many readers may already have in their repertoires. More importantly, we have concentrated on developing the basic tools of the subject, in order to familiarize the student with current methodology. Thus we focus on results which can be proved from a common point of view and steer away from miraculous arguments which can be used only once. In this spirit, our treatment is deliberately not encyclopedic, but is rather aimed at what we see as the major threads and key topics of current interest.

It is our hope that this book can be read by a student without the benefit of a course or an instructor. To encourage this possibility, we have tried to include details when they might have been omitted, and to discuss the motivations for proceeding as we do. Moreover, we have woven an extensive selection of exercises into the text. These exercises are particularly designed to give the novice some experience and familiarity with both the material and the tools being developed.

One of the fundamental differences between the theories of commutative and noncommutative rings is that the former arise naturally as rings of functions, whereas the latter arise naturally as rings of operators. For example, early in the twentieth century, some of the first noncommutative rings that received serious study were certain rings of differential operators. More generally, given any set of linear transformations of a vector space, we can form the algebra of linear transformations generated by this set, and many problems of interest concerning the original transformations become module-theoretic questions, where we view the original vector space as a module over the algebra we have created. In many modern applications, in turn, it is essential to regard noncommutative rings as rings of transformations or operators of various kinds. We are partial to this point of view. This has led us to emphasize the role of modules when studying a ring, for modules are simply ways of representing the ring at issue in terms of endomorphisms of abelian groups. Also, when defining a ring we have tended to present it as a ring of operators of some sort rather than by taking a more formal approach, such as giving generators and relations. For example, when constructing rings of fractions, we have preferred to find them as rings of endomorphisms rather than as sets of equivalence classes of ordered pairs of elements.

Although the noetherian condition is very natural in commutative ring theory, since it holds for the rings of integers in algebraic number fields and for the coordinate rings crucial to algebraic geometry, it was originally less clear that this condition would be useful in the noncommutative setting. For instance, Jacobson's definitive book of 1956 makes only minimal mention of noetherian rings. Similarly, prime ideals, essential in the commutative theory, seemed to have relatively less importance for noncommutative rings; in fact, because of the fundamental role of representation-theoretic ideas in the development of the noncommutative theory, the initial emphasis in the subject was almost exclusively on irreducible representations (i.e., simple modules) and primitive ideals (i.e., annihilators of irreducible representations). In the meantime, however, it has turned out that various important types of noncommutative rings – in particular, certain infinite group rings and the enveloping algebras of finite dimensional Lie algebras – are in fact noetherian. This has been used to good effect in recent work on the representation theory of the corresponding groups and Lie algebras, just as the theory of finite dimensional algebras and artinian rings has played a key role in research on the representations of finite groups. Also, as soon as noetherian rings and their modules received serious attention, prime ideals forced themselves into the picture, even in contexts where the original interest had been entirely in primitive ideals. As a consequence, we have made prime ideals a major theme in our text.

The first important result in the theory of noncommutative noetherian rings was proved relatively recently, in 1958. This was Goldie's Theorem, which gives an analog of a field of fractions for factor rings R/P where R is a noetherian ring and P a prime ideal of R. Once this milestone had been reached, noetherian ring theory proceeded apace, partly from its own impetus and partly through feedback from neighboring areas in which noetherian ideas found applications. One of our aims in this book has been to develop those aspects of the theory of noetherian rings which have the strongest connections with the representation-theoretic areas to which we have alluded. However, as these areas have their own extensive theories, it was impossible to treat them in any generality in this volume. Instead, we present a brief discussion in the prologue, giving some representative examples to which the theory in the text can be applied relatively directly, without extensive side trips into technical intricacies.

To give the reader an idea of the historical sources of the theory, we have included some bibliographical notes at the end of each chapter. We have sought to make these notes as accurate as possible, but as with any evolving theory complete precision is difficult to attain, especially since in many research papers sources are not well documented. Some inaccuracies are thus probably inevitable, and we apologize in advance for any that may have occurred.

In an appendix we discuss some open problems in noetherian ring theory; we hope that our readers will be stimulated to solve them.

For helpful comments on various drafts of the book, we would like to thank A. D. Bell, K. A. Brown, D. A. Jordan, T. H. Lenagan, P. Perkins, L. W. Small, and J. T. Stafford. We would also like to thank our competitors J. C. McConnell and J. C. Robson for letting us see early drafts of various chapters from their noetherian rings book [1987].

Prologue

Since much of the current interest in noncommutative noetherian rings stems from applications of the general theory to several specific types, we present here a very sketchy introduction to some major areas of application: polynomial identity rings, group algebras, rings of differential operators, enveloping algebras, and quantum groups. Each of these areas has a very extensive theory of its own, far too voluminous to be incorporated into a book of this size. (See for instance Rowen [1980], Passman [1985], McConnell-Robson [2001], and Brown-Goodearl [2002]). Instead, we shall concentrate on surrogates – some classes of rings that are either simple prototypes or analogs of the major types just mentioned – which we can investigate by relatively direct methods while still exhibiting the flavor of the areas they represent. These surrogates are module-finite algebras over commutative rings (for polynomial identity rings), skew-Laurent rings (for group algebras), formal differential operator rings (for rings of differential operators and some enveloping algebras), and general skew polynomial rings (for some enveloping algebras and quantum groups). They will be introduced below and studied in greater detail in the following two chapters.

We will conclude the Prologue with a few comments about our notation and terminology.

• POLYNOMIAL IDENTITY RINGS •

Commutativity in a ring may be phrased in terms of a relation that holds identically, namely $xy - yx = 0$ for all choices of x and y from the ring. More complicated identities sometimes also hold in noncommutative rings. For example, if x and y are any 2×2 matrices over a commutative ring S, then the trace of $xy - yx$ is zero, and so it follows from the Cayley-Hamilton Theorem that $(xy - yx)^2$ is a scalar matrix. Consequently, $(xy - yx)^2$ commutes with every 2×2 matrix z, and hence the relation

$$(xy - yx)^2 z - z(xy - yx)^2 = 0$$

holds for all choices of x, y, z from the ring $M_2(S)$ (the ring of all 2×2 matrices over S). A much deeper result, the Amitsur-Levitzki Theorem, asserts that, for all choices of $2n$ matrices $x_1, \ldots, x_{2n}$ from the $n \times n$ matrix ring $M_n(S)$,

$$\sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(2n)} = 0,$$

where S_{2n} is the symmetric group on $\{1, 2, \ldots, 2n\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of a permutation σ (namely $+1$ or -1, depending on whether σ is even or odd).

Such an "identical relation" on a ring may be thought of as saying that a certain polynomial – with noncommuting variables! – vanishes identically on the ring. In this context, the polynomials are usually restricted to having integer coefficients. Thus a *polynomial identity* on a ring R is a polynomial $p(x_1, \ldots, x_n)$ in noncommuting variables $x_1, \ldots, x_n$ with coefficients from $\mathbb{Z}$ such that $p(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$. A *polynomial identity ring*, or *P.I. ring* for short, is a ring R which satisfies some *monic* polynomial identity $p(x_1, \ldots, x_n)$ (that is, among the monomials of highest total degree which appear in p, at least one has coefficient 1).

The Amitsur-Levitzki Theorem implies that every matrix ring over a commutative ring is a P.I. ring, and consequently so is every factor ring of a subring of such a matrix ring. For example, the endomorphism ring of a finitely generated module A over a commutative ring S has this form. To see that, identify A with S^n/K for some $n \in \mathbb{N}$ and some submodule K of S^n, and identify the matrix ring $M_n(S)$ with the endomorphism ring of S^n. Then the set

$$T = \{f \in M_n(S) \mid f(K) \subseteq K\}$$

is a subring of $M_n(S)$, the set $I = \{f \in M_n(S) \mid f(S^n) \subseteq K\}$ is an ideal of T, and $T/I \cong \operatorname{End}_S(A)$. Therefore $\operatorname{End}_S(A)$ is a P.I. ring.

Certain algebras over commutative rings fit naturally into this context. Recall that an *algebra* over a commutative ring S is just a ring R equipped with a specified ring homomorphism ϕ from S to the center of R. (The map ϕ is not assumed to be injective.) Then ϕ is used to define products of elements of S with elements of R: For $s \in S$ and $r \in R$, we set sr and rs equal to $\phi(s)r$ (or $r\phi(s)$, which is the same because $\phi(s)$ is in the center of R). Using this product, we can view R as an S-module. We say that R is a *module-finite* S-algebra if R is a finitely generated S-module. Note that $R \cong \operatorname{End}_R(R_R) \subseteq \operatorname{End}_S(R)$ as rings, and so any polynomial identity satisfied in $\operatorname{End}_S(R)$ will also be satisfied in R. Taking the preceding paragraph into account, we conclude that any module-finite algebra over a commutative ring is a P.I. ring.

The class of module-finite algebras over commutative noetherian rings provides us with a supply of prototypical examples of noetherian P.I. rings. To illustrate some applications of the noetherian theory to P.I. rings, we shall at times work out consequences of the former for our class of examples. In this setting, we will be able to replace P.I. theory by some much more direct methods from commutative ring theory.

• GROUP ALGEBRAS •

One of the earliest stimuli to the modern development of noncommutative ring theory came from the study of *group representations*. The key idea was to study a group G by "representing" it in terms of linear transformations on a vector space V, namely, by studying a group homomorphism ϕ from G to the group of invertible linear transformations on V. Linear algebra can then be used to study the group $\phi(G)$, and the information gleaned can be pulled back to G via the *representation* ϕ. Using ϕ, there is an "action" of G on V, namely, a product $G \times V \to V$ given by the rule $g{\cdot}v = \phi(g)(v)$, and since ϕ is a homomorphism, $(gh){\cdot}v = g{\cdot}(h{\cdot}v)$ for all $g, h \in G$ and $v \in V$. This looks a lot like module multiplication, if we ignore the lack of an addition for elements of G, and in fact V is called a *G-module* in this situation.

To make V into an actual module over a ring, we build G and its multiplication into a ring, along with whichever field k we are using for scalars. Just make up a vector space with a basis which is in one-to-one correspondence with the elements of G, identify each element of G with the corresponding basis element, and then extend the multiplication from G to this vector space linearly:

$$\Big(\sum_{g\in G} \alpha_g g\Big)\Big(\sum_{h\in G} \beta_h h\Big) = \sum_{g,h\in G} (\alpha_g\beta_h)(gh).$$

The result is a k-algebra called the *group algebra* of G over k, denoted $k[G]$ or just kG. Except for the obvious changes in terminology, $k[G]$-modules are the same as representations of G on vector spaces over k.

In the case of a finite group G, the group algebra $k[G]$ is finite dimensional, and the theory of finite dimensional algebras has much to say about representations of G. A noetherian group algebra is known to occur when G is *polycyclic-by-finite*, that is, when G has a series of subgroups

$$G_0 = (1) \subset G_1 \subset \cdots \subset G_n \subseteq G_{n+1} = G$$

such that each G_{i-1} is a normal subgroup of G_i and G_i/G_{i-1} is infinite cyclic for $i = 1, \dots, n$, while G/G_n is finite. (It is an open problem whether $k[G]$ is noetherian only when G is polycyclic-by-finite.) One of the simplest infinite non-abelian examples is the group G with two generators x, y and the sole relation $yxy^{-1} = x^{-1}$. In this case, elements of $k[G]$ can all be put in the form $\sum_{i=-n}^{n} p_i(x)y^i$, where each $p_i(x)$ is a Laurent polynomial (i.e., a polynomial in x and x^{-1}). From the relation $yxy^{-1} = x^{-1}$ it follows that $yp(x)y^{-1} = p(x^{-1})$ for all Laurent polynomials $p(x)$. Hence, the Laurent polynomial ring $k[x, x^{-1}]$ is sent into itself by the map $p(x) \mapsto yp(x)y^{-1}$, and this map coincides with the map $p(x) \mapsto p(x^{-1})$, which is an automorphism of $k[x, x^{-1}]$.

The pattern of this example suggests a construction that starts with a ring R and an automorphism α of R, and then builds a ring T whose elements look like Laurent polynomials over R in a new indeterminate y, except that instead of commuting with y, elements $r \in R$ satisfy the relation $yry^{-1} = \alpha(r)$, or $yr = \alpha(r)y$. Since the usual multiplication of polynomials has been "skewed" through α, the ring T is called a *skew-Laurent ring*. Thus the group algebra of the previously discussed group with the relation $yxy^{-1} = x^{-1}$ may be viewed as a skew-Laurent ring with coefficient ring $k[x, x^{-1}]$.

We shall see that any skew-Laurent ring with a noetherian coefficient ring is itself noetherian. This fact actually provides the method used to show that the group algebra of any polycyclic-by-finite group G is noetherian. Namely, if

$$G_0 = (1) \subset G_1 \subset \cdots \subset G_n \subseteq G_{n+1} = G$$

is the series of subgroups of G occurring in the definition of "polycyclic-by-finite," it can be shown that for $i = 1, \ldots, n$ the group algebra $k[G_i]$ is isomorphic to a skew-Laurent ring whose coefficient ring is $k[G_{i-1}]$. Starting at the bottom with $k[G_0] = k$, it follows immediately by induction that $k[G_n]$ is noetherian. It then just remains to observe that $k[G]$ is a finitely generated right or left module over $k[G_n]$ to conclude that $k[G]$ itself is noetherian. In particular, we see from this discussion that (iterated) skew-Laurent rings are a better match for group algebras of polycyclic-by-finite groups than might have been suggested by the very special example given above.

• RINGS OF DIFFERENTIAL OPERATORS •

Another early stimulus to noncommutative ring theory came from the study of differential equations. Late in the nineteenth century, it was realized that, just as polynomial functions provide a useful means of dealing with algebraic equations, "differential operators" are convenient for handling linear differential equations. For example, a homogeneous linear differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

can be rewritten very compactly as $d(y) = 0$, where d denotes the *linear differential operator*

$$a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x).$$

From this viewpoint, d is a linear transformation on some vector space of functions, and the solution space of the original differential equation is just the null space of d.

To be a bit more specific, let us consider the special case in which coefficients and solutions are real-valued rational functions. Then our differential operators are $\mathbb{R}$-linear transformations on the field $\mathbb{R}(x)$. The composition of two differential operators is certainly a linear transformation, but it takes a minute to see that such a composition is actually another differential operator. In order to make the notation more convenient, we use the symbol D

to denote the operator d/dx. If we form the operator composition Da, which means "first multiply by the function $a(x)$ and then differentiate," we see that

$$(Da)(y) = (ay)' = ay' + a'y = aD(y) + a'y$$

for any function y, and so $Da = aD + a'$. Iterated use of this identity then allows us to write the composition of any two differential operators in the standard form of a differential operator, i.e., as a sum of terms $a_i D^i$, where $a_i \in \mathbb{R}(x)$. Thus the collection of differential operators on $\mathbb{R}(x)$ forms a ring, which is sometimes denoted $B_1(\mathbb{R})$. We may think of $B_1(\mathbb{R})$ as a polynomial ring $\mathbb{R}(x)[D]$ in which, however, the multiplication is twisted to make a noncommutative ring. This ring attracted particular attention early in the twentieth century, when it was proved that it is a principal ideal domain (that is, all left and right ideals are principal) and that it satisfies a form of unique factorization.

We can of course proceed in the same way using for coefficients other rings of functions that are closed under differentiation. For example, if we start with the ring $\mathbb{C}[x]$ of complex polynomials, the ring of differential operators we obtain looks like a twisted polynomial ring in two variables, $\mathbb{C}[x][D]$. This ring is called the *first complex Weyl algebra* and is denoted $A_1(\mathbb{C})$. More generally, we may start with a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ in several variables and build differential operators using the partial derivatives $\partial/\partial x_i$, abbreviated D_i. This results in a twisted polynomial ring in $2n$ variables, $\mathbb{C}[x_1, \ldots, x_n][D_1, \ldots, D_n]$, which is called the *n-th complex Weyl algebra* and is denoted $A_n(\mathbb{C})$.

Examples such as $B_1(\mathbb{R})$ and $A_n(\mathbb{C})$, which will often recur in the text, can be taken as representative of a more general class that has assumed some importance in recent years: rings of differential operators on algebraic varieties. We cannot discuss these in detail but will content ourselves with indicating how they can be described. We recall that a *complex affine algebraic variety* is a subset V of $\mathbb{C}^n$ which is the set of common zeroes of some collection I of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. If I contains all the polynomials that vanish on V, then I is an ideal in the polynomial ring, and the factor ring $R = \mathbb{C}[x_1, \ldots, x_n]/I$ is the *coordinate ring of* V. The *ring of differential operators on* V, denoted $\mathcal{D}(V)$, consists of those differential operators on $\mathbb{C}[x_1, \ldots, x_n]$ that induce operators on R, modulo those that induce the zero operator on R. More precisely, the set

$$S = \{s \in A_n(\mathbb{C}) \mid s(I) \subseteq I\}$$

is a subring of $A_n(\mathbb{C})$, the set

$$J = \{s \in A_n(\mathbb{C}) \mid s(\mathbb{C}[x_1, \ldots, x_n]) \subseteq I\}$$

is an ideal of S, and $\mathcal{D}(V) = S/J$. It has been proved that $\mathcal{D}(V)$ is noetherian in case V has no singularities and in case V is a curve, but it appears that for higher dimensional varieties with singularities $\mathcal{D}(V)$ is usually not noetherian.

• ENVELOPING ALGEBRAS •

A *Lie algebra* over a field k is a vector space L over k equipped with a nonassociative product $[\cdot\cdot]$ satisfying the usual bilinear and distributive laws as well as the rules

$$[xx] = 0 \qquad \text{and} \qquad [x[yz]] + [y[zx]] + [z[xy]] = 0$$

for all $x, y, z \in L$. For example, $\mathbb{R}^3$ equipped with the usual vector cross product is a real Lie algebra. The standard model for the product in a Lie algebra is the *additive commutator* operation $[x, y] = xy - yx$ in an associative ring (more precisely, any associative k-algebra when equipped with the operation $[\cdot,\cdot]$ becomes a Lie algebra over k). Conversely, starting with a Lie algebra L, one can build an associative k-algebra $U(L)$ using the elements of L as generators, together with relations $xy - yx = [xy]$ for all $x, y \in L$. The algebra $U(L)$ is called the *(universal) enveloping algebra* of L, and it is known to be noetherian in case L is finite dimensional. (Whether it is possible for the enveloping algebra of an infinite dimensional Lie algebra to be noetherian is an open problem.)

The simplest Lie algebra L with a nonzero product is 2-dimensional, with a basis $\{x, y\}$ such that $[yx] = x$. Elements of the enveloping algebra $U(L)$ can in that case all be put into the form $\sum_{i=0}^{n} p_i(x)y^i$, where each $p_i(x)$ is an ordinary polynomial in the variable x. In $U(L)$, the relation $[yx] = x$ becomes $[y, x] = x$, and from this it follows easily that $[y, p(x)] = x\frac{d}{dx}(p(x))$ for all polynomials $p(x)$. In other words, $[y, -]$ maps the polynomial ring $k[x]$ into itself, and its action on polynomials is given by the operator $x\frac{d}{dx}$. The reader should note that this is very similar to the ring $A_1(\mathbb{C})$ discussed above, the difference being that in $A_1(\mathbb{C})$ we have the relation $[D, p(x)] = \frac{d}{dx}(p(x))$. (In fact, $U(L)$ in our example is isomorphic to the subalgebra of $A_1(k)$ generated by x and xD.)

Abstracting this pattern, we may start with a ring R and a map $\delta : R \to R$ which is a *derivation* (that is, δ is additive and satisfies the usual product rule for derivatives) and then build a larger ring T using an indeterminate y such that $[y, r] = \delta(r)$ for all $r \in R$. The elements of T look like differential operators $\sum r_i\delta^i$ on R, except that it may be possible for $\sum r_i\delta^i$ to be the zero operator without all the coefficients r_i being zero. Thus, the elements $\sum r_i y^i$ in T are called *formal differential operators*, and T is called a *(formal) differential operator ring*.

We shall see that all formal differential operator rings with noetherian coefficient rings are themselves noetherian, and we shall view them as representative analogs of enveloping algebras. The analogy is actually a little better than one might think, knowing only the single example mentioned above. Namely, if L is a finite dimensional Lie algebra which can be realized as a Lie algebra of upper triangular matrices over k (using $[\cdot,\cdot]$ for the Lie product), then $U(L)$ can be built as an *iterated differential operator ring* through a series of extensions

$$T_0 = k \subset T_1 \subset \cdots \subset T_m = U(L),$$

where each T_i is isomorphic to a differential operator ring with coefficients from T_{i-1}. (Over $\mathbb{C}$, the finite dimensional Lie algebras that can be realized as upper triangular matrices are precisely the *solvable* Lie algebras.)

Among the most important Lie algebras are the *special linear* Lie algebras $\mathfrak{sl}_n(k)$, which consist of $n \times n$ matrices over k having trace 0 (again with Lie product $[\cdot,\cdot]$). In particular, $\mathfrak{sl}_2(k)$ is 3-dimensional, and one typically chooses the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as a basis. The Lie products (commutators) among e, f, and h are given by

$$[ef] = h \qquad [he] = 2e \qquad [hf] = -2f.$$

In the enveloping algebra $U(\mathfrak{sl}_2(k))$, the Lie product relation $[he] = 2e$ becomes $he - eh = 2e$, or $eh = (h-2)e$. It follows that $ep(h) = p(h-2)e$ for any polynomial $p(h) \in k[h]$. This allows us to think of the k-algebra generated by e and h as a twisted polynomial ring in two variables, $k[h][e]$, where the twist arises from the map $p(h) \mapsto p(h-2)$. The latter map being an automorphism of $k[h]$, we thus see that $k[h][e]$ is a polynomial version of the skew-Laurent ring construction discussed above.

When the element f is added to the picture, we have to deal with the relations $ef - fe = h$ and $hf - fh = -2f$, or $fe = ef - h$ and $fh = (h+2)f$. The last equation is reminiscent of (the inverse of) the automorphism $p(h) \mapsto p(h-2)$ above, and indeed there is an automorphism α of the ring $k[h][e]$ such that $\alpha(h) = h + 2$ and $\alpha(e) = e$. The relation $fe = ef - h$ turns out to be accounted for by a linear map δ on $k[h][e]$ such that $\delta(e) = -h$ and $\delta(h) = 0$, the end result being $fr = \alpha(r)f + \delta(r)$ for all $r \in k[h][e]$. (The map δ is similar to a derivation – it satisfies a "skew product rule" $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$ and is called a *skew derivation*.) We thus view $U(\mathfrak{sl}_2(k))$ as a twisted polynomial ring in three variables, $k[h][e][f]$, where the final twist involves both an automorphism and a skew derivation. Each of the steps $k \rightsquigarrow k[h] \rightsquigarrow k[h][e] \rightsquigarrow k[h][e][f]$ is a type of *skew polynomial ring*, and in summary we say that $U(\mathfrak{sl}_2(k))$ is an *iterated skew polynomial ring*. For our purposes, this structure is a means to let us see that $U(\mathfrak{sl}_2(k))$ is a noetherian domain. (While $U(\mathfrak{sl}_n(k))$ is a noetherian domain for any n, other methods are needed to prove that, since $U(\mathfrak{sl}_n(k))$ is not an iterated skew polynomial ring when $n \geq 3$.)