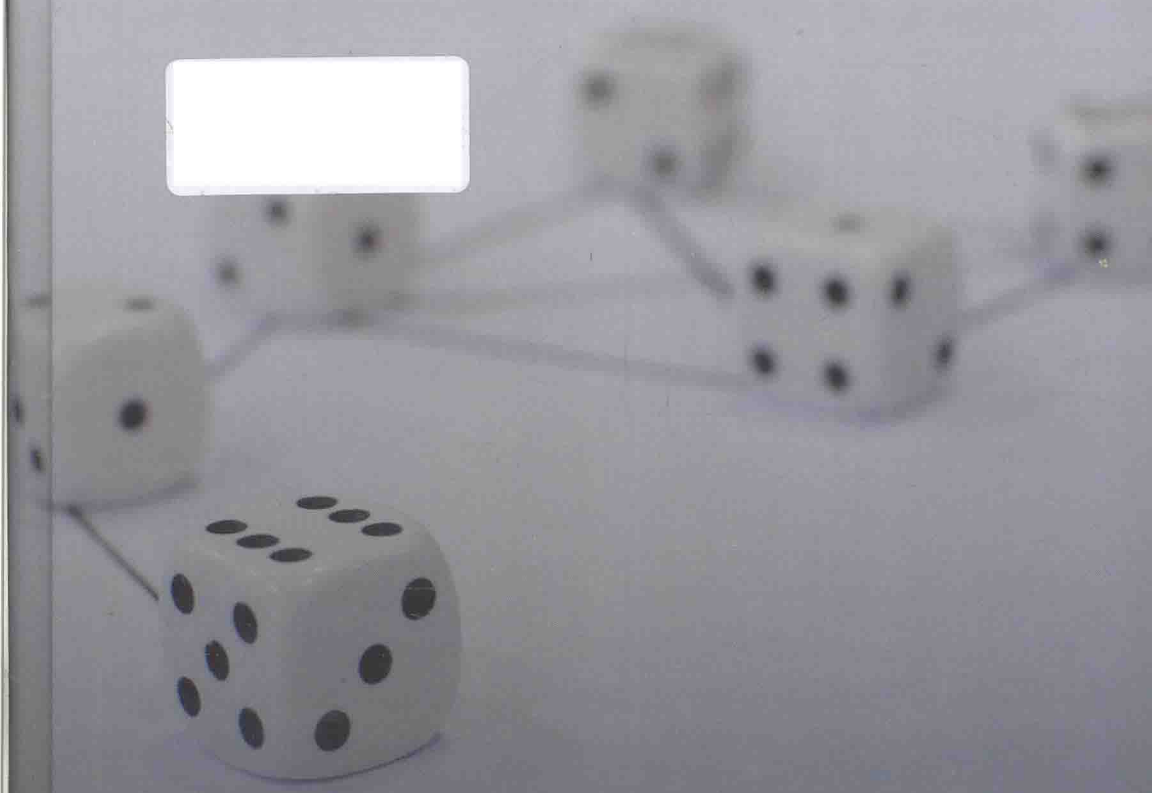

INTRODUCTION TO

# RANDOM GRAPHS

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# Introduction to Random Graphs

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# Introduction to Random Graphs

From social networks such as Facebook, the World Wide Web and the Internet, to the complex interactions between proteins in the cells of our bodies, we constantly face the challenge of understanding the structure and development of networks. The theory of random graphs provides a framework for this understanding, and in this book the authors give a gentle introduction to the basic tools for understanding and applying the theory. Part one includes sufficient material, including exercises, for a one-semester course at the advanced undergraduate or beginning graduate level. The reader is then well prepared for the more advanced topics in Parts two and three. A final part provides a quick introduction to the background material needed.

All those interested in discrete mathematics, computer science or applied probability and their applications will find this an ideal introduction to the subject.

**Alan Frieze** is a Professor in the Department of Mathematical Sciences at Carnegie Mellon University. He has authored more than 300 publications in top journals and was invited to be a plenary speaker at the Seoul ICM 2014. In 1991 he received the Fulkerson prize in discrete mathematics.

**Michał Karoński** is a Professor in the Departments of Mathematics and Computer Science at Adam Mickiewicz University and Emory University. He is founder of the Discrete Mathematics group in Poznań and since 1990 has served as co-Editor-in-Chief of *Random Structures and Algorithms*.

To Carol and Jola

# Preface

Our purpose in writing this book is to provide a gentle introduction to a subject that is enjoying a surge in interest. We believe that the subject is fascinating in its own right, but the increase in interest can be attributed to several factors. One factor is the realization that networks are "everywhere." From social networks such as Facebook, the World Wide Web and the Internet to the complex interactions between proteins in the cells of our bodies, we face the challenge of understanding their structure and development. By and large natural networks grow in an unpredictable manner and this is often modeled by a random construction. Another factor is the realization by Computer Scientists that NP-hard problems are often easier to solve than their worst-case suggests and that an analysis of running times on random instances can be informative.

## History

Random graphs were used by Erdős [274] to give a probabilistic construction of a graph with large girth and large chromatic number. It was only later that Erdős and Rényi began a systematic study of random graphs as objects of interest in their own right. Early on they defined the random graph $\mathbb{G}_{n,m}$ and founded the subject. Often neglected in this story is the contribution of Gilbert [367] who introduced the model $\mathbb{G}_{n,p}$, but clearly the credit for getting the subject started goes to Erdős and Rényi. Their seminal series of papers [275], [277], [278], [279] and, in particular, [276] on the evolution of random graphs laid the groundwork for other mathematicians to become involved in studying properties of random graphs.

In the early eighties the subject was beginning to blossom and it received a boost from two sources. First was the publication of the landmark book of Béla Bollobás [130] on random graphs. Around the same time, the

Discrete Mathematics group at Adam Mickiewicz University began a series of conferences in 1983. This series continues biennially to this day and is now a conference attracting more and more participants.

The next important event in the subject was the start of the journal *Random Structures and Algorithms* in 1990 followed by *Combinatorics, Probability and Computing* a few years later. These journals provided a dedicated outlet for work in the area and are flourishing today.

## Scope of the book

We have divided the book into four parts. Part one is devoted to giving a detailed description of the main properties of $\mathbb{G}_{n,m}$ and $\mathbb{G}_{n,p}$. The aim is not to give best possible results, but instead to give some idea of the tools and techniques used in the subject, as well as to display some of the basic results of the area. There is sufficient material in Part one for a one-semester course at the advanced undergraduate or beginning graduate level. Once one has finished the content of the first part, one is equipped to continue with material of the remainder of the book, as well as to tackle some of the advanced monographs such as Bollobás [130] and the more recent one by Janson, Łuczak and Ruciński [432].

Each chapter comes with a few exercises. Some are fairly simple and these are designed to give the reader practice with making some of the estimations that are so prevalent in the subject. In addition, each chapter ends with some notes that lead through references to some of the more advanced important results that have not been covered.

Part two deals with models of random graphs that naturally extend $\mathbb{G}_{n,m}$ and $\mathbb{G}_{n,p}$. Part three deals with other models. Finally, in Part four, we describe some of the main tools used in the area along with proofs of their validity.

Having read this book, the reader should be in a good position to pursue research in the area and we hope that this book will appeal to anyone interested in Combinatorics or Applied Probability or Theoretical Computer Science.

## Acknowledgements

Several people have helped with the writing of this book and we would like to acknowledge their help. First there are the students who have sat in on courses based on early versions of this book and who helped to iron out the many typo's etc.

We would next like to thank the following people for reading parts of the book before final submission: Andrew Beveridge, Deepak Bal, Malgorzata Bednarska, Patrick Bennett, Mindaugas Bloznelis, Antony Bonato, Boris Bukh, Fan Chung, Amin Coja-Oghlan, Colin Cooper, Andrzej Dudek, Asaf Ferber, Nikolas Fountoulakis, Catherine Greenhill, Dan Hefetz, Paul Horn, Hsien–Kuei Hwang, Jerzy Jaworski, Tony Johansson, Mihyun Kang, Michael Krivelevich, Tomasz Łuczak, Colin McDiarmid, Andrew McDowell, Hosam Mahmoud, Mike Molloy, Tobias Müller, Rajko Nenadov, Wesley Pegden, Boris Pittel, Dan Poole, Pawel Prałat, Andrzej Ruciński, Katarzyna Rybarczyk, Matas Šileikis, Greg Sorkin, Joel Spencer, Dudley Stark, Angelika Steger, Prasad Tetali, Andrew Thomason, Linnus Wästlund, Nick Wormald and Stephen Young.

Thanks also to Béla Bollobás for his advice on the structure of the book.

## Conventions/Notation

Often in what follows, we give an expression for a large positive integer. It might not be obvious that the expression is actually an integer. In which case, the reader can rest assured that he/she can round up or down and obtained any required property. We avoid this rounding for convenience and for notational purposes.

In addition we list the following notation:

### Mathematical relations

- $f(x) = O(g(x))$: $|f(x)| \leq K|g(x)|$ for some constant $K > 0$ and all $x \in \mathbf{R}$.
- $f(x) = \Theta(g(x))$: $f(n) = O(g(x))$ and $g(x) = O(f(x))$.
- $f(x) = o(g(x))$ as $x \to a$: $f(x)/g(x) \to 0$ as $x \to a$.
- $A \ll B$: $A/B$ is sufficiently small for the succeeding arguments.
- $A \gg B$: $A/B$ is sufficiently large for the succeeding arguments.
- $A \approx B$: $A/B \to 1$ as some parameter converges to 0 or $\infty$ or another limit.
- $[n]$: This is $\{1,2,\dots,n\}$. In general, if $a < b$ are positive integers, then $[a,b] = \{a,a+1,\dots,b\}$.
- If $S$ is a set and $k$ is a non-negative integer then $\binom{S}{k}$ denotes the set of $k$-element subsets of $S$. In particular, $\binom{[n]}{k}$ dnotes the set of $k$-sets of $\{1,2,\dots,n\}$.

### Graph Notation

- $G = (V,E)$: $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.
- $e(G)$: $|E(G)|$.

- $N(S) = N_G(S)$ where $S \subseteq V(G)$. $\{w \notin S : \exists v \in S \text{such that } \{v,w\} \in E\}$.
- For a graph $H$, $aut(H)$ denotes the number of automorphisms of $H$.

**Random Graph Models**

- $[n]$: The set $\{1,2,\ldots,n\}$.
- $\mathcal{G}_{n,m}$: The family of all labeled graphs with vertex set $V = [n] = \{1,2,\ldots,n\}$ and exactly $m$ edges.
- $\mathbb{G}_{n,m}$: A random graph chosen uniformly at random from $\mathcal{G}_{n,m}$.
- $E_{n,m} = E(\mathbb{G}_{n,m})$.
- $\mathbb{G}_{n,p}$: A random graph on vertex set $[n]$ where each possible edge occurs independently with probability $p$.
- $E_{n,p} = E(\mathbb{G}_{n,p})$.
- $\mathbb{G}_{n,m}^{\delta \geq k}$: $G_{n,m}$, conditioned on having minimum degree at least $k$.
- $\mathbb{G}_{n,n,p}$: A random bipartite graph with vertex set consisting of two disjoint copies of $[n]$ where each of the $n^2$ possible edges occurs independently with probability $p$.
- $\mathbb{G}_{n,r}$: A random $r$-regular graph on vertex set $[n]$.
- $\mathcal{G}_{n,\mathbf{d}}$: The set of graphs with vertex set $[n]$ and degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.
- $\mathbb{G}_{n,\mathbf{d}}$: A random graph chosen uniformly at random from $\mathcal{G}_{n,\mathbf{d}}$.
- $\mathbb{H}_{n,m;k}$: A random $k$-uniform hypergraph on vertex set $[n]$ and $m$ edges of size $k$.
- $\mathbb{H}_{n,p;k}$: A random $k$-uniform hypergraph on vertex set $[n]$ where each of the $\binom{n}{k}$ possibles edge occurs independently with probability $p$.
- $\vec{\mathbb{G}}_{k-out}$: A random digraph on vertex set $[n]$ where each $v \in [n]$ independently chooses $k$ random out-neighbors.
- $\mathbb{G}_{k-out}$: The graph obtained from $\vec{\mathbb{G}}_{k-out}$ by ignoring orientation and coalescing multiple edges.

**Probability**

- $\mathbb{P}(A)$: The probability of event $A$.
- $\mathbb{E}Z$: The expected value of random variable $Z$.
- $h(Z)$: The entropy of random variable Z.
- $Po(l)$: A random variable with the Poisson distribution with mean $l$.
- $N(0,1)$: A random variable with the normal distribution, mean 0 and variance 1.
- $\text{Bin}(n,p)$: A random variable with the binomial distribution with parameters $n$, the number of trials and $p$, the probability of success.
- $EXP(l)$: A random variable with the exponential distribution, mean $l$ i.e. $\mathbb{P}(EXP(l) \geq x) = e^{-lx}$. We sometimes say *rate* $1/l$ in place of mean $l$.

- w.h.p.: A sequence of events $\mathscr{A}_n, n = 1, 2, \ldots$, is said to occur *with high probability* (w.h.p.) if $\lim_{n\to\infty} \mathbb{P}(\mathscr{A}_n) = 1$.
- $\xrightarrow{D}$: We write $X_n \xrightarrow{D} X$ to say that a random variable $X_n$ *converges in distribution* to a random variable $X$, as $n \to \infty$. Occasionally we write $X_n \xrightarrow{D} N(0,1)$ (resp. $X_n \xrightarrow{D} Po(l)$) to mean that $X$ has the corresponding normal (resp. Poisson) distribution.

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