MATHEMATICAL NOTES

HODGE THEORY

Edited by
Eduardo Cattani, Fouad El Zein,
Phillip A. Griffiths, and Lê Dũng Tráng

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Hodge Theory

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Preface

Between 14 June and 2 July 2010, the *Summer School on Hodge Theory and Related Topics* and a related conference were hosted by the ICTP in Trieste, Italy. The organizers of the conference were E. Cattani, F. El Zein, P. Griffiths, Lê D. T., and L. Göttsche. Attending the summer school were a large and diverse group of young mathematicians, including students, many of whom were from countries in the developing world. The conference brought together leading active researchers in Hodge theory and related areas.

In the summer school, the lectures were intended to provide an introduction to Hodge theory, a subject with deep historical roots and one of the most vibrant areas in contemporary mathematics. Although there are a number of excellent texts treating various aspects of Hodge theory, the subject remains quite difficult to learn due in part to the breadth of background needed. An objective of the summer school was to give an exposition of basic aspects of the subject in an accessible framework and language, while presenting—in both the school and the conference—selected topics in Hodge theory that are at the forefront of current research. These dual goals are reflected in the contents of this volume, many of whose entries do in fact present both the fundamentals and the current state of the topics covered.

The lectures by Eduardo Cattani on Kähler manifolds provide a lucid and succinct account of the basic geometric objects that give rise to a Hodge structure in cohomology. Smooth projective complex algebraic varieties are Kähler manifolds, and the Hodge structures on their cohomology provide an extremely rich set of invariants of the variety. Moreover, they are the fundamental building blocks for the mixed Hodge structure in the cohomology of general complex algebraic varieties. The realization of cohomology by harmonic forms provides the connection between analysis and geometry, and the subtle implications of the Kähler condition on harmonic forms are explained in these lectures.

In modern Hodge theory, one of the most basic tools is the algebraic de Rham theorem as formulated by Grothendieck. This result, whose origins date from the work of Picard and Poincaré, relates the complex analytic and algebraic approaches to the topology of algebraic varieties via differential forms. The contribution to the volume by Fouad El Zein and Loring Tu presents a new treatment of this subject, from the basic sheaf-theoretic formulation of the classical de Rham theorem through the statement and proof of the final result.

Mixed Hodge structures are the subject of the lectures by El Zein and Lê Dung Trang. According to Deligne, the cohomology of a general complex algebraic variety carries a functorial mixed Hodge structure, one that is built up from the pure Hodge structures on smooth projective varieties which arise when the original variety is compactified and its singularities are resolved. This is a far-reaching, deep, and complex topic, one whose essential aspects are addressed with clarity in their presentation.

The subject of period domains and period maps, covered in the lectures by Jim Carlson, involves Lie-theoretic and differential geometric aspects of Hodge theory. Among the items discussed is the topic of infinitesimal period relations (transversality) and its curvature implications, which are fundamental to the analysis of the limiting mixed Hodge structures that reflect the behavior of the Hodge structures on the cohomology of smooth varieties as those varieties acquire singularities.

The lectures by Luca Migliorini and Mark de Cataldo on the Hodge theory of maps deal with the Hodge-theoretic aspects of arbitrary proper maps between general algebraic varieties. The fundamental result here, the decomposition theorem, shows how variations of mixed Hodge structures combine with intersection cohomology to describe the deep Hodge-theoretic properties of the above maps. This subject is complementary to the lectures by Patrick Brosnan and El Zein on variations of mixed Hodge structures; between the two, they provide the framework for Morihiko Saito's unifying theory of mixed Hodge modules.

One of the very basic aspects of Hodge theory is the analysis of how the Hodge structure associated to a smooth algebraic variety varies when the algebraic variety degenerates to a singular one in an algebraic family. The fundamental concept here is that of a limiting mixed Hodge structure. The analysis of how Hodge structures degenerate is given in the lectures by Eduardo Cattani, a presentation that covers the subject from its origins up through many of its most recent and deepest aspects.

The subject of the lectures by Brosnan and El Zein is variations of mixed Hodge structures. This topic brings together the material from Cattani's lectures on variation of Hodge structure and those by El Zein and Lê on mixed Hodge structures. As the title suggests, it describes how the mixed Hodge structures vary in an arbitrary family of algebraic varieties. The treatment given here presents a clear and efficient account of this central aspect of Hodge theory.

One of the original purposes of Hodge theory was to understand the geometry of algebraic varieties through the study of the algebraic subvarieties that lie in it. A central theme is the interplay between Hodge theory and the Chow groups, and the conjectures of Hodge and of Bloch–Beilinson serve to frame much of the current work in this area. These topics are covered, from the basic definitions to the forefront of current research, in the five lectures by Jacob Murre. The lectures by Mark Green are a continuation of the topic of algebraic cycles as introduced in the lectures of Murre. Hodge theory again appears because it provides—conjecturally—the basic invariants and resulting structure of the group of cycles. In these lectures, the Hodge theory associated to a new algebraic variety, whose function field is generated by the coefficients in the defining equation of the original variety, is introduced as a method for studying algebraic cycles, one that conjecturally reduces the basic open questions to a smaller and more fundamental set. It also suggests how one may reduce many questions, such as the Hodge conjecture, to the case of varieties defined over number fields.

A principal unsolved problem in Hodge theory is the Hodge conjecture, which provides a Hodge-theoretic characterization of the fundamental cohomology classes supported by algebraic cycles. For an algebraic variety defined over an abstract field, the Hodge conjecture implies that the Hodge-theoretic criterion for a cohomology class to support an algebraic cycle is independent of the embedding of that field into the complex numbers. Hodge classes with this property are called "absolute." The basic known result here is due to Deligne: it states that all Hodge classes on abelian varieties are absolute. This theorem relates the arithmetic and complex analytic aspects of these varieties, and a self-contained proof of it is presented in the article by François Charles and Christian Schnell.

Arithmetic automorphic representation theory is one of the most active areas in current mathematical research, centering around what is known as the Langlands program. The basic methods for studying the subject are the trace formula and the theory of Shimura varieties. The latter are algebraic varieties that arise from Hodge structures of weight 1; their study relates the analytic and arithmetic aspects of the algebraic varieties in question. The lectures by Matt Kerr provide a Hodge-theoretic approach to the study of Shimura varieties, one that is complementary to the standard, more algebraic, presentations of the subject.

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