Set Theory and the Number Systems

the
Number Systems

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SET THEORY AND THE NUMBER SYSTEMS

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To
Walter

Preface

The word "mathematics" is derived from the Greek τὰ μαθηματικά, which means "learning" or "knowledge." Today, "mathematics" is applied not to knowledge in general but to knowledge of a particular kind. Still, there is some misunderstanding as to exactly what sort of knowledge is involved. Often, the layman thinks of mathematics as arithmetic, and of the mathematician as one who is very quick and clever at performing computations. The scientist may think of mathematics as another science of which arithmetic is only a very small part. Most professional mathematicians, however, regard their subject as neither of these things. (Indeed, it is commonly said, by mathematicians, that they can seldom add two numbers correctly or balance a bank statement.) Present-day mathematicians generally regard their subject as the study of logical consequences of systems of axioms, where the axioms are completely arbitrary statements made about completely abstract objects. Thus, one could argue that mathematics is more closely related to philosophy or the arts than to the sciences.

The purpose of this book is to acquaint the reader with some of the basic concepts and techniques of mathematics, and to illustrate how these concepts and techniques may be applied to an axiomatic development of various number systems and to a study of cardinal numbers. Along the way, many properties of the real numbers and of the cardinal numbers will be investigated.

The book is intended for a one-semester course. Such a course might be taken by students who are not planning to specialize in mathematics but who, nevertheless, wish to learn about the nature of mathematics; by students who do plan to specialize in mathematics and want a foundation for their later studies; by prospective mathematics teachers; and by those preparing to teach "modern" mathematics. The book is also intended for the general reader—whether he be a curious adult or a curious high-school or college student.

Very little previous knowledge of mathematics—at most, some elementary algebra—is required for an understanding of this book. Exercises are provided at the ends of various sections. The reader is strongly urged to carry out these exercises so that he may have adequate preparation for handling the ideas that follow.

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1-1. INTRODUCTION

In order to understand some of the examples in the first two chapters, one should be familiar with the following notions, which are developed more thoroughly in Chapters 3 through 6.

1. The numbers $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$ are called integers. An even integer is an integer which is a multiple of 2, that is, a number which can be expressed in the form $2n$, where n is an integer. The even integers are $\cdots, -4, -2, 0, 2, 4, \cdots$. An odd integer is a number which can be expressed in the form $2n + 1$, where n is an integer. The odd integers are $\cdots, -3, -1, 1, 3, 5, \cdots$. The integers $1, 2, 3, \cdots$ are called positive integers. The integers $\cdots, -3, -2, -1$ are called negative integers. The integer 0 is neither positive nor negative. An integer a is said to be divisible by an integer b, where b is not 0, if there is an integer c such that $a = bc$; that is, a is the product of b and c. For example, 6 is divisible by 2, but 2 is not divisible by 6.

2. Numbers of the form $\frac{p}{q}$, where p and q are integers and q is not 0, are called rational numbers (ratios of integers). For example, $\frac{3}{4}$, $\frac{-2}{27}$, and $\frac{10}{-9}$ are rational numbers. Moreover, any integer n is a rational number, since n may be expressed in the form $\frac{n}{1}$.

3. Numbers such as $\sqrt{2}$, $\sqrt[3]{2}$, and π, which cannot be expressed as ratios of integers, are called irrational numbers.

4. All the rational numbers together with all the irrational numbers constitute the set of real numbers or the real number system.

If x is a real number, then x^2 denotes the product of x with x, x^3 denotes the product of x^2 with x, and so on.

For any two real numbers m and n, either m is greater than n, m equals n, or m is less than n. If m is greater than n, we write $m > n$; if

m equals n, we write $m = n$; if m is less than n, we write $m < n$. For example, $5 > 3$; $3 = 3$; $3 < 5$. The notation $m \geqq n$ or $m \leqq n$ indicates that m is greater than or equal to n, or that m is less than or equal to n, respectively; for example, $5 \geqq 3$; $5 \geqq 5$; $3 \leqq 5$; $5 \leqq 5$. The notation $3 \leqq x \leqq 7$ indicates that x is a number between 3 and 7 and that x may equal 3 or x may equal 7. The notation $3 < x < 7$ indicates that x is a number between 3 and 7 but that x may not equal 3 or 7. The notation $3 \leqq x < 7$ indicates that x is a number between 3 and 7 and that x may equal 3 but not 7.

The symbol ∞, called infinity, does not denote a real number. To assert that $0 < x < \infty$ is to assert that x is a real number which is greater than 0; to assert that $-\infty < x < \infty$ is to assert that x is any real number.

The absolute value of a real number x, denoted by $|x|$, is defined as follows:

$$|x| = \begin{cases} x \text{ if } x \geqq 0; \\ -x \text{ if } x < 0. \end{cases}$$

For example, $|5| = 5$; $|-5| = 5$; $|0| = 0$.

5. A complex number is a number of the form $a + bi$, where a and b are real numbers and i is a number (not real) with the property that $i^2 = -1$. For example, $2 + 3i$, $5 - 2i$, and $6i$ are complex numbers. Any real number is also a complex number, with b equal to 0.

1-2. AXIOMS, THEOREMS, AND VALIDITY

In the logical development of any branch of mathematics, certain undefined (or primitive) concepts must be introduced in order to avoid circular definitions. As an everyday example, suppose that someone wishes to learn the meaning of the word "parasang," and in his dictionary he reads that a parasang is a farsakh. Never having heard of the word "farsakh" before, he consults his dictionary again only to read that a farsakh is a parasang. Surely our poor man has learned nothing from these (circular) definitions. In geometry, "point" and "line" often are taken as undefined concepts, for defining a point as the intersection of two distinct lines and a line as a join of two distinct points clearly is a meaningless procedure. Indeed, with a little thought, one will be convinced that it is impossible to define *every* word in a language.

After certain concepts have been chosen as undefined, all additional concepts will be precisely defined in terms of these undefined ones.

A mathematical statement is a declarative sentence about one or more undefined concepts.

Examples 1. "Two points determine a line" is a mathematical statement.

2. "Construct a line joining two given points" is not a mathematical statement.

Certain mathematical statements are designated as axioms. Axioms are assumptions, and they state *everything* that is assumed to be known about the undefined concepts. A statement is considered valid if it is an axiom or if it can be proved from the axioms and definitions, in accordance with the rules of logical deduction, to be described in Sec. 1-4. A proof of a statement consists of reasoning by which one justifies the validity of the statement from the axioms and definitions.

It is important to note that the validity of a statement is completely unrelated to the truth of the statement. Since the axioms are unproved statements about undefined concepts, it certainly can make no sense to speak of the truth of the axioms, let alone of any statement derived from them. Bertrand Russell once said[1] that mathematics may be defined as the subject in which we never know what we are talking about or whether what we are saying is true. This is a sound observation for, since the basic concepts are undefined, one does not know what he is talking about when he speaks of these concepts, and, since the validity of a statement depends upon unproved axioms, one can never know whether the statement is true. Indeed, "true" can have no meaning when applied to a statement containing undefined concepts.

A theorem is a valid statement of importance; that is, an important logical consequence of the axioms or definitions. A corollary is a valid statement which is an immediate consequence of, or a special case of, a preceding definition or theorem. A lemma is a valid statement which is proved preliminary to proving a theorem. It may be of interest only in that it is needed to establish a certain theorem.

A theorem (or corollary or lemma) may be expressed in the form, "If P, then Q," where P and Q are statements.

Examples

1. If two angles of a triangle are equal, then the sides opposite the angles are equal.
2. If an integer a is less than an integer b, then $a + 1$ is less than or equal to b.

Here, the statement "P" is called the hypothesis, the word "if" indicating that P is assumed to be valid. The statement "Q" is called the conclusion, the word "then" indicating that Q must be proved valid in order to prove the validity of the theorem. The axioms are understood to be assumed throughout the development of the mathematical theory and,

[1] "Recent Work on the Principles of Mathematics," *International Monthly*, Vol. 4 (1901), p. 84.

hence, generally are not repeated in the hypothesis. Because of this, a theorem (or corollary or lemma) may simply have the form "Q."

Example. The sum of the lengths of two sides of a triangle is greater than the length of the third side.

In this case, the conclusion only is stated, and the theorem asserts that the conclusion holds, providing that the axioms hold.

EXERCISE

State whether each of the following is a mathematical statement and give all reasons for your answers.

a) 6 is greater than 2.
b) 2 is greater than 6.
c) What is a number?
d) Why is 6 greater than 2?
e) Multiplying by 66,548,989.237 is a lot of work.
f) Curses on these numbers with many digits.

1-3. MATHEMATICAL USAGE OF CERTAIN WORDS

Certain words are used frequently in mathematical discourse. We shall now consider some of these words and how their use in mathematics differs from their everyday use. In order to give concrete examples of some of the notions introduced, in this discussion we shall assume some of the elementary properties of the real numbers.

The capital letters P, Q, R, and so on, will be used to denote statements. For example, P might be the statement "$1 = 1$" or the statement "$1 + 1 = 2$."

The word "and" may be placed between two statements "P" and "Q" to form a new statement "P and Q." The statement "P and Q" is valid only when both P and Q are valid.

Example. Let P be the statement "2 is an even number." Let Q be the statement "5 is an even number." Then "P and Q" is the statement "2 is an even number and 5 is an even number," which is not valid, even though P is.

The word "or" may be placed between two statements "P" and "Q" to form a new statement "P or Q." In mathematics, this compound statement is understood to mean that either P is valid or Q is valid and *possibly both are valid*. Thus, "or" is used in the "and/or" sense (one or the other or both). One should contrast this with everyday usage, where "or" frequently means one or the other, but *not* both.

Examples 1. Let P be the statement "4 is an even number." Let Q be the statement "5 is an even number." Then "P or Q" is the statement "4 is an even number or 5 is an even number," which is valid, since P is.

2. Let P be the statement "4 is an even number." Let Q be the statement "6 is an even number." Then "P or Q" is the statement "4 is an even number or 6 is an even number," which is valid, since both P and Q are. (In everyday usage, one might argue that "P or Q" is not valid simply because *both* P and Q are valid.)

The word "not" may be placed before a statement "P" to form a new statement "Not P." This new statement is called the negation of P.

Example. Let P be the statement "4 is an even number." Then "Not P" is the statement "Not 4 is an even number" or "4 is not an even number."

The negation of the compound statement "P and Q" is "Not P or not Q," for to deny that P and Q are both valid is to assert that at least one of them is not valid.

The negation of the compound statement "P or Q" is "Not P and not Q," for to deny that either P or Q is valid is to assert that both P and Q are not valid.

The negation of the statement "Not P" is "Not (not P)"; that is, "P."

The word "equals," denoted symbolically by $=$, may be placed between two expressions to indicate that the two expressions are names of the same identical object. Hence, if we know that $a = a'$, then we also know that $a' = a$, since, if a and a' denote the same object, then a' and a denote the same object. By the same argument, if $a = a'$ and $a' = a''$, then $a = a''$, for all three expressions are names of the same object. If two expressions a and b are *not* equal, we write "$a \neq b$."

Mathematical statements often concern sets of objects rather than a single object. A quantifier is a word or phrase telling how many (quant-) of the objects of the set are involved in the statement.

The quantifiers "each," "any," "every," and "all" may be used interchangeably.

Example. Each integer is a number; any integer is a number; every integer is a number; all integers are numbers.

The word "any" may sometimes be used in a way in which the other three words cannot. For example, suppose that we want to prove that for all real numbers x, x^2 is positive. We might start by saying, "Let x be any real number," indicating that we are choosing an unspecified real number x in an arbitrary fashion. Then we would proceed to show that x^2 is positive, where x is our arbitrary unspecified number. From this, we would conclude that all real numbers have positive squares, since our arbitrarily chosen number does. The word "any," used in this way, cannot be replaced by "each," "every," or "all."

The quantifiers "some" and "there exist" may be used interchangeably.

Example. For some number x, $x^2 = 1$; there exists a number x such that $x^2 = 1$.

Consider the statement "All integers are numbers." To deny this statement, one would assert "There exists an integer x such that x is not a number." To deny the statement "There exists an integer x such that x is not a number," one would assert "All integers are numbers."

If P and Q are any two statements, one may form the compound statement "If P, then Q," or its equivalent, "P implies Q." This is denoted symbolically by $P \Longrightarrow Q$.

The statement "$Q \Longrightarrow P$" is called the converse of the statement "$P \Longrightarrow Q$." The converse of a valid statement need not be valid.

Examples

1. Consider the statement "If an integer is divisible by 6, then it is divisible by 3." Its converse is "If an integer is divisible by 3, then it is divisible by 6." The original statement is valid; the converse is not.
2. Consider the statement "If a triangle is equilateral, then it is equiangular." Its converse is "If a triangle is equiangular, then it is equilateral." Both the original statement and its converse are valid.

The statement "Not $Q \Longrightarrow$ not P" is called the contrapositive of the statement "$P \Longrightarrow Q$." It will be shown, in Sec. 1-4, that if a statement is valid, then its contrapositive is valid, and, conversely, if the contrapositive of a statement is valid, then the statement itself is valid.

Consider the statement "P and $Q \Longrightarrow S$." Its converse is "$S \Longrightarrow P$ and Q." Its contrapositive is the statement "Not $S \Longrightarrow$ not (P and Q)"; that is, "Not $S \Longrightarrow$ not P or not Q."

If P implies Q, then P is said to be a sufficient condition for Q, for, if P is valid, then Q must be valid also. That is, the validity of P is sufficient for the validity of Q. Similarly, it is said that Q is a necessary condition for P, for, if "Not Q" were valid, then "Not P" would be valid also. That is, it is necessary for Q to be valid in order for P to be valid.