ORDINARY DIFFERENTIAL EQUATIONS IN THE COMPLEX DOMAIN

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PREFACE

A friend recently wrote: "Mathematics is the creation of flesh and blood, not just novelty-curious and wise automatons. We ought to devote some part of our efforts to increasing understanding of the observable universe." This book is a contribution to these efforts. We praise famous men, men who created beautiful structures and directed the course of mathematics. A quotation from the *Edda* may be appropriate: "One thing I know that never dies, the call after a dead man."

The structures that the masters built are not just beautiful to the eye; they are also eminently useful. In these days when the need is felt for "applicable mathematics" and "utilitas mathematica," it is fitting to recall that few domains of mathematics are so widely applicable as the theory of ordinary differential equations. This range of ideas is dear to my heart: for close to 60 years much of my time has been given to the cultivation of differential equations.

The book deals with ordinary differential equations in the complex domain. It covers the usual ground, more or less. Here and there features are introduced that are less canonical. There is a general emphasis on growth questions: the dominants and minorants of Section 2.7 constitute a variation of the majorant theme. The Nevanlinna theory of value distribution plays an important role: it is applied to the Malmquist-Wittich-Yosida theorem (Sections 4.5 and 4.6) and to Boutroux's investigations (Sections 11.2 and 12.3). The Papperitz-Wirtinger account of Riemann's lectures on hypergeometric functions and their uniformization by elliptic modular functions has been rescued from oblivion (Section 10.5). Finally, the second half of Chapter 12 presents the Emden-Fowler and the Thomas-Fermi equations, quadratic systems, and Russell Smith's recent work on polynomial autonomous systems, all matters of some novelty.

The reader is expected to have some knowledge of complex variables, a subject in which our students are frequently weak: they comprehend little, and often their knowledge is too abstract and is of the wrong kind. Elementary manipulative skill is too often atrophied. Hence the second half of Chapter 1 of this book is devoted to complex analysis. Chapter 11 has an appendix on elliptic functions, and modular and theta functions are discussed at some length in Sections 7.3 and 10.5. These sections should help the reader.

Each chapter has a list of references to the literature, and there is a bibliography at the end of the book. The exercises at the ends of sections comprise some 675 items.

The book was written at the behest of Harry Hochstadt, who scoffed at my misgivings and attempts to escape; he has aided and abetted my efforts, and I owe him hearty thanks. May the book live up to his expectations. Thanks are also due to numerous friends who have helped with advice, bibliographical and biographical information, and constructive criticism. Specific mention should be made of L. V. Ahlfors, O. Borůvka, W. N. Everitt, C. Frymann, Ih-Ching Hsu, S. Kakutani, Z. Nehari, D. Rosenthal, I. Schoenberg, R. Smith, H. Wittich, C. C. Yang and K. Yosida. J. A. Donaldson and H. Hochstadt have kindly helped with the proofreading. Further, I am grateful to Addison-Wesley Publishing Co. and to the R. Society of Edinburgh for permission to use copyrighted material. I am also indebted to the Department of Mathematics of the University of California at San Diego for Xerox copying and to the National Science Foundation for support (Grant GP 41127). Finally, I owe much to my family, wife and sons, for encouragement, help, interest, and patience.

EINAR HILLE

La Jolla, California
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CONTENTS

1
INTRODUCTION

In this chapter we shall list, with or without proof, various facts which will be used in the following. They will fall under two general headings: (I) algebraic and geometric structures, and (II) analytic structures. Under I we shall remind the reader of abstract spaces, metrics, linear vector spaces, norms, fixed point theorems, functional inequalities, partial ordering, linear transformations, matrices, algebras, etc. Under II we discuss analytic functions: analyticity, Cauchy's integral, Taylor and Maclaurin series, entire and meromorphic functions, power series, growth, analytic continuation, and permanency of functional equations. This is quite an ambitious program, and the reader may find the density of ideas per page somewhat overwhelming. He is advised to skim over the pages in the first reading and to return to the relevant material as, if, and when needed.

I. Algebraic and Geometric Structures

1.1. VECTOR SPACES

The term *abstract space* is often used as a synonym for *set* or *point set*, but the term usually indicates that the author intends to endow the set with an algebraic or geometric structure or both. If a Euclidean space R^n serves as a prototype of a space, we obtain an abstract space by abstracting (= withdrawing) some of its properties while keeping others. Incidentally, *property* is an undefined term (we can obviously not use the definition ascribed to Jean Jacques Rousseau: La propriété c'est le vol!). We denote our space by **X** and its elements by **x**, **y**, **z**, . . . We say that the space has an algebraic structure if one or more algebraic operations can be performed on the elements, or if a notion of order is meaningful at least for some elements.

The set is a *linear vector space* if the operations of *addition* and *scalar multiplication* can be performed. It is required that the set be an Abelian group under addition, that is, $\mathbf{x}+\mathbf{y}$ is defined as an element of $\mathbf{X}$, addition is associative and commutative, there is a unique *neutral element* $\mathbf{0}$ such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for all $\mathbf{x}$, and every element $\mathbf{x}$ has a unique *negative*, $-\mathbf{x}$, with $\mathbf{x}+(-\mathbf{x})=\mathbf{0}$.

To define scalar multiplication we need a field of scalars, which is almost always taken to be the real field R or the complex field C. For any scalar α and element $\mathbf{x}$ there is a unique element $\alpha\mathbf{x}$; scalar multiplication is associative, it is distributive with respect to addition, and $1\cdot\mathbf{x}=\mathbf{x}$, where 1 is the unit element of the scalars.

We speak of a *real* or a *complex vector space* according as the scalar field is R or C. The elements of $\mathbf{X}$ are now called *vectors.* A linear vector space which also contains the product of any two of its elements is called an *algebra.* The set of all polynomials in a variable t is obviously an algebra, and so is the set of all functions $t\mapsto f(t)$ which are continuous at a point t_0.

Consider a set of n vectors $\mathbf{x}_j$ in $\mathbf{X}$, and let the underlying scalar field be denoted by F. Then the vectors $\mathbf{x}_j$ are *linearly independent* over F if

$$\alpha_1\mathbf{x}_1+\alpha_2\mathbf{x}_2+\cdots+\alpha_n\mathbf{x}_n=\mathbf{0} \tag{1.1.1}$$

implies that all the α's are zero. They are *linearly dependent* over F if multipliers α_j can be found so that (1.1.1) holds with $|\alpha_1|+|\alpha_2|+\cdots+|\alpha_n|>0$. Emphasis should be placed on "over F," for restricting F to a subfield F^0 or extending it to a larger field F^* affects the independence relations. Thus 1 and $2^{1/2}$ are linearly independent over Q, the field of rational numbers, but not over A, the field of algebraic numbers. The space $\mathbf{X}$ is said to be of *dimension* n if it contains a set of n linearly independent vectors while any $n+1$ vectors are linearly dependent. It is of infinite dimension if n linearly independent vectors can be found for any n.

The notion of *partial ordering* is another form of algebraic structure. We say that $\mathbf{X}$ is *partially ordered* if for some pairs $\mathbf{x}$, $\mathbf{y}$ of $\mathbf{X}$ there is an ordering relation $\mathbf{x}\leqslant\mathbf{y}$ (equivalently, $\mathbf{y}\geqslant\mathbf{x}$) which is *reflexive, proper,* and *transitive,* that is, (i) $\mathbf{x}\leqslant\mathbf{x}$ for all $\mathbf{x}$, (ii) $\mathbf{x}\leqslant\mathbf{y}$ and $\mathbf{y}\leqslant\mathbf{x}$ imply $\mathbf{x}=\mathbf{y}$, (iii) $\mathbf{x}\leqslant\mathbf{y}$, $\mathbf{y}\leqslant\mathbf{z}$ imply $\mathbf{x}\leqslant\mathbf{z}$. If $\mathbf{X}$ is linear as well as partially ordered, we should have

$$\mathbf{x}\leqslant\mathbf{y} \quad \text{implies} \quad \mathbf{x}+\mathbf{a}\leqslant\mathbf{y}+\mathbf{a} \quad \text{for all } \mathbf{a}, \tag{1.1.2}$$

$$\mathbf{x}\leqslant\mathbf{y} \quad \text{implies} \quad \alpha\mathbf{x}\leqslant\alpha\mathbf{y} \quad \text{for } \alpha>0. \tag{1.1.3}$$

In this case $\mathbf{X}$ has a *positive cone* $\mathbf{X}^+$, defined as the set of all elements $\mathbf{x}\in\mathbf{X}$ such that $\mathbf{0}\leqslant\mathbf{x}$. This positive cone is invariant under addition and multiplication by positive scalars. It contains **0**, the *neutral element*, usually referred to as the *zero element*. We now have $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \in \mathbf{X}^+$.

The set of real valued continuous functions on the closed interval [0, 1], say $C[0, 1]$, is an algebra. We define its positive cone $\mathbf{X}^+$ as the set of functions $t \mapsto f(t)$ whose values on [0, 1] are nonnegative. We have $f \leq g$ if $g(t) - f(t)$ is nonnegative in [0, 1].

A more prosaic example may be helpful: the fowl in a hen-yard are partially ordered under the pecking order.

EXERCISE 1.1

1. Consider the space of all polynomials $P(t)$ in a real variable which take on real values. Show that **X** is an algebra.
2. An order relation $P < Q$ is established in **X** by defining P as positive if its values are positive for all large positive values of t. Show that this ordering is a *trichotomy* in the sense that for a given P there are only three possibilities: (i) P is positive, (ii) $-P$ is positive, or (iii) $P = 0$.
3. An order relation is said to be *Archimedean* if $\mathbf{x} < \mathbf{y}$ implies the existence of an integer n such that $\mathbf{y} < n\mathbf{x}$. (The natural ordering of the reals is Archimedean.) Show that the order defined in Problem 2 is non-Archimedean inasmuch as the elements fall into rank classes R_k, where R_k consists of all polynomials of exact degree k, each R_k is Archimedean, but if f is a positive element of R_j and g a positive element of R_k with $j < k$, then $f < g$ and $nf < g$ for all n. Verify.
4. Prove that $1, t, t^2, \ldots, t^n$ are linearly independent over $\mathbb{R}$. What is the dimension of the space formed by these elements?

1.2. METRIC SPACES

A *metric space* is one in which there is defined a notion of *distance* subject to the following conditions:

D_1. For any pair of points P and Q of **X** a number $d(P, Q) \geq 0$ is defined, called the distance from P to Q such that $d(P, Q) = 0$ iff $P = Q$.

D_2. $d(P, Q) = d(Q, P)$.

D_3. For any R we have $d(P, Q) \leq d(P, R) + d(R, Q)$.

These notions go back to the work in the 1890's of Hermann Minkowski (1864–1909) on what he called the "geometry of numbers." He was chiefly concerned with the extremal properties of linear and of quadratic forms, for which he found alternative definitions of distance, adjusted to the problem in hand. Minkowski did not always require D_2. Condition D_3 is the *triangle inequality*.

We say that a linear vector space is *normed* if the following conditions hold:

N_1. For each $\mathbf{x} \in \mathbf{X}$ there is assigned a number $\|\mathbf{x}\| \geq 0$ such that $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$.

N_2. $\|\alpha \mathbf{x}\| = |\alpha| \, \|\mathbf{x}\|$ for each α in the scalar field.

N_3. $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

A normed linear vector space becomes a metric space by setting

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|. \tag{1.2.1}$$

In a metric space we can do analysis since the fundamental operation of analysis, that of finding limits of a sequence, becomes meaningful. If $\{\mathbf{x}_n\}$ is a sequence in the metric space $\mathbf{X}$, we say that $\mathbf{x}_n$ converges to $\mathbf{x}_0$ and

$$\mathbf{x}_0 = \lim \mathbf{x}_n \quad \text{if} \quad \lim_{n\to\infty} d(\mathbf{x}_0, \mathbf{x}_n) = 0. \tag{1.2.2}$$

We say that $\{\mathbf{x}_n\}$ is a *Cauchy sequence* if, given any $\epsilon > 0$, there exists an N such that

$$d(\mathbf{x}_m, \mathbf{x}_n) < \epsilon \qquad \text{for } m, n > N. \tag{1.2.3}$$

If (1.2.2) holds, it follows that $\{\mathbf{x}_n\}$ is a Cauchy sequence, but the converse is not necessarily true, for there may be gaps in the space. A metric space $\mathbf{X}$ is said to be *complete* if all Cauchy sequences converge to elements of the space. Euclidian spaces are complete, and so are various function spaces that will be encountered in the following. The space $\mathbf{Q}$ of rational numbers is not complete.

Various notions of real analysis are meaningful in complete metric spaces, such as the concepts of *closure*, *open set*, *closed set*, and ϵ-neighborhood. The Bolzano-Weierstrass theorem need not be valid in a complete metric space, that is, there may be bounded infinite point sets without a limit point. Incidentally, "bounded" means that the set can be enclosed in a "sphere" $d(\mathbf{x}, \mathbf{0}) < R$. The *topological diameter* $d(S)$ of a subset of $\mathbf{X}$ is the least upper bound of the distances $d(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}$ and $\mathbf{y}$ in S.

EXERCISE 1.2

1. The Euclidean norm $\|\mathbf{x}\|_2$ of $\mathbf{x}$ in $\mathbf{C}^n$ is $[\Sigma_{j=1}^n |x_j|^2]^{1/2}$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and the x's are complex numbers. Alternative norms are $\|\mathbf{x}\|_1 = \Sigma_{j=1}^n |x_j|$ and $\|\mathbf{x}\|_\infty = \sup |x_j|$. Show that they are indeed acceptable norms. Between what limits do they lie if $\|\mathbf{x}\|_2 = 1$?
2. How do you define an open set in these three normed topologies? Show that a set open in one of them is also open with respect to the others. Verify that $\mathbf{C}^n$ is complete in all three metrics.

3. Let $\mathbf{X} = C[0, 1]$ be the set of all functions, $t \mapsto f(t)$, continuous in the closed interval $[0, 1]$. Define a Cauchy sequence if $\|f\| = \sup_{0 \le t \le 1} |f(t)|$, and show that the space is complete.

1.3. MAPPINGS

We shall study mappings from a metric space $\mathbf{X}$ into a metric space $\mathbf{Y}$, both being complete. We shall often have $\mathbf{Y} = \mathbf{X}$.

The mapping T is a pairing of points $\mathbf{x}$ of $\mathbf{X}$ with points $\mathbf{y}$ of $\mathbf{Y}$, say $\langle \mathbf{x}, \mathbf{y} \rangle$. Here to every $\mathbf{x}$ of $\mathbf{X}$ is ordered a unique $\mathbf{y}$ of $\mathbf{Y}$. To $\mathbf{x}_1 \neq \mathbf{x}_2$ correspond the two values $\mathbf{y}_1$ and $\mathbf{y}_2$, which may or may not be distinct, in fact every $\mathbf{x} \in \mathbf{X}$ may be mapped on the same point $\mathbf{y}_0 \in \mathbf{Y}$. The mapping is *onto* (a *surjection* in the Bourbaki language) if every point of $\mathbf{Y}$ is the image $\mathbf{y}$ of at least one $\mathbf{x}$ in $\mathbf{X}$. It is $(1, 1)$ (read "one to one") if

$$\mathbf{x}_1 \neq \mathbf{x}_2 \quad \text{implies} \quad \mathbf{y}_1 = T(\mathbf{x}_1) \neq T(\mathbf{x}_2) = \mathbf{y}_2. \tag{1.3.1}$$

The mapping is *bounded* if there exists a finite M such that

$$d[T(\mathbf{x}_1), T(\mathbf{x}_2)] \leq M\, d(\mathbf{x}_1, \mathbf{x}_2). \tag{1.3.2}$$

This is a generalized *Lipschitz condition* and implies continuity of $T(\mathbf{x})$ with respect to $\mathbf{x}$.

If $\mathbf{X}$ and $\mathbf{Y}$ are linear vector spaces over the same scalar field, and if

$$T(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2) = \alpha_1 T(\mathbf{x}_1) + \alpha_2 T(\mathbf{x}_2), \tag{1.3.3}$$

then T is called a *linear transformation.* It is bounded iff

$$d[T(\mathbf{x}), \mathbf{0}] \leq M\, d(\mathbf{x}, \mathbf{0}). \tag{1.3.4}$$

The most important case is that in which $\mathbf{X}$ and $\mathbf{Y}$ are complete normed linear vector spaces, in which case the spaces are called *Banach spaces* after the Polish mathematician Stefan Banach (1892–1945), who termed them B-spaces. In this case (1.3.4) takes the form

$$\|T[\mathbf{x}]\| \leq M \|\mathbf{x}\|. \tag{1.3.5}$$

If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$ and the transformation is $(1, 1)$ if

$$T(\mathbf{x}) = \mathbf{0} \quad \text{implies} \quad \mathbf{x} = \mathbf{0}.$$

If $\mathbf{X}$ and $\mathbf{Y}$ are B-spaces, the set $\mathbf{E}(\mathbf{X}, \mathbf{Y})$ of linear bounded transformations on $\mathbf{X}$ to $\mathbf{Y}$ is also a B-space under the norm

$$\|T\| = \sup \|T[\mathbf{x}]\|, \tag{1.3.6}$$

the supremum being taken with respect to all elements $\mathbf{x}$ of $\mathbf{X}$ of norm 1. The algebraic operations in $\mathbf{E}(\mathbf{X}, \mathbf{Y})$ are defined in the obvious manner by

$$[T_1 + T_2][\mathbf{x}] = T_1[\mathbf{x}] + T_2[\mathbf{x}], \qquad (\alpha T)[\mathbf{x}] = \alpha T[\mathbf{x}]. \tag{1.3.7}$$

If $\mathbf{Y} = \mathbf{X}$, we write $\mathbf{E}[\mathbf{X}]$ for $\mathbf{E}[\mathbf{X}, \mathbf{X}]$ and note that products are definable in the obvious manner by

$$(T_1 T_2)[\mathbf{x}] = T_1(T_2[\mathbf{x}]). \tag{1.3.8}$$

This gives

$$\|T_1 T_2\| \leq \|T_1\| \|T_2\|. \tag{1.3.9}$$

It may be shown that $\mathbf{E}(\mathbf{X}, \mathbf{Y})$ and $\mathbf{E}(\mathbf{X})$ are complete in the normad metric, so they are B-spaces. Also, $\mathbf{E}(\mathbf{X})$, which is a normed algebra, actually is a B-algebra since it is a B-space and satisfies (1.3.9).

If $T \in \mathbf{E}(\mathbf{X})$ and is $(1, 1)$, there is an *inverse transformation* T^{-1} such that

$$T^{-1}[T[\mathbf{x}]] = \mathbf{x}, \quad \forall \mathbf{x}, \qquad T[T^{-1}[\mathbf{y}]] = \mathbf{y} \quad \text{if } \mathbf{y} = T[\mathbf{x}]. \tag{1.3.10}$$

EXERCISE 1.3

1. Show that $\mathbf{E}(\mathbf{C}^n)$ is complete. Use any of the metrics for $\mathbf{C}^n$ listed in Problem 1.2: 1.
2. If T is a linear transformation, verify that $T(\mathbf{0}) = \mathbf{0}$. Here on the right stands the zero element of $\mathbf{Y}$, while on the left we operate on the zero element of $\mathbf{X}$. [*Hint*: $T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0})$.]
3. If T is $(1, 1)$, why does $T(\mathbf{x}) = \mathbf{0}$ imply $\mathbf{x} = \mathbf{0}$ and vice versa?
4. Why is (1.3.6) a norm? Show that it is the least value that M can have in (1.3.5).
5. Prove (1.3.9).

1.4. LINEAR TRANSFORMATIONS ON $\mathbf{C}^n$ INTO ITSELF; MATRICES

The simplest of all linear transformations are those which map $\mathbf{C}^n$ into itself. If T is such a transformation, then T is uniquely determined by linearity and its effect on the *basis* of $\mathbf{C}^n$. Any set of n linearly independent vectors would serve as a basis, but we may just as well use the unit vectors

$$\mathbf{e}_j = (\delta_{jk}), \tag{1.4.1}$$

where δ_{jk} is the Kronecker delta, that is, the vector whose jth component is one, all others being zero. This gives

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n, \tag{1.4.2}$$

if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in the coordinate system defined by the vectors $\mathbf{e}_j$. Now

T takes vectors into vectors, so there are n^2 complex numbers a_{jk} such that

$$T[\mathbf{e}_k] = a_{1k}\mathbf{e}_1 + a_{2k}\mathbf{e}_2 + \cdots + a_{nk}\mathbf{e}_n, \quad k = 1, 2, \ldots, n. \tag{1.4.3}$$

The linearity of T then gives

$$T[\mathbf{x}] = T\left(\sum_{k=1}^{n} x_k \mathbf{e}_k\right) = \sum_{k=1}^{n} x_k T[\mathbf{e}_k]$$

or

$$T[\mathbf{x}] = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{jk}x_k\right)\mathbf{e}_j \equiv \mathbf{y}, \tag{1.4.4}$$

from which we can read off the components of the vector $\mathbf{y}$.

The quadratic array

$$\mathscr{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{1.4.5}$$

is known as a *matrix*—more precisely, the matrix of the transformation T with respect to the chosen basis. We can now write T symbolically as

$$\mathbf{y} = T[\mathbf{x}] = \mathscr{A} \cdot \mathbf{x}, \tag{1.4.6}$$

where the last member may be considered as the product of the matrix $\mathscr{A}$ with the column vector $\mathbf{x}$, the result being the column vector $\mathbf{y}$.

We have to decide when the mapping defined by T is (1, 1). Here the condition $T[\mathbf{x}] = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$ now takes the form that the homogeneous system

$$\sum a_{jk}x_k = 0, \quad j = 1, 2, \ldots, n, \tag{1.4.7}$$

must have the unique solution

$$x_1 = x_2 = \cdots = x_n = 0.$$

This will happen as long as

$$\det(\mathscr{A}) \neq 0. \tag{1.4.8}$$

In this case the mapping is also onto, since for a given vector $\mathbf{y}$ we can solve the system

$$\sum_{k=1}^{n} a_{jk}x_k = y_j, \quad j = 1, 2, \ldots, n, \tag{1.4.9}$$

uniquely for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. It follows that T has a unique inverse, also an element of $\mathbf{E}[\mathbf{C}^n]$, i.e., a linear bounded transformation of $\mathbf{C}^n$ into itself. With this transformation goes a matrix $\mathcal{A}^{-1}$, which we refer to as the *inverse* of $\mathcal{A}$. The fact that its elements may be computed from (1.4.9) shows that the element in the place (j, k) is A_{kj}/Δ, where A_{jk} is the cofactor of a_{jk} in the determinant $\Delta = \det(\mathcal{A})$.

We can define algebraic operations and a norm in the set $\mathfrak{M}_n$ of n-by-n matrices in terms of which the set becomes a Banach algebra. This follows from the fact that there is a (1, 1) correspondence between the linear transformations T in $\mathbf{E}(\mathbf{C}^n)$ and their matrices. Then to $T_1 + T_2$, αT, and $T_1 T_2$ correspond

$$(a_{jk} + b_{jk}) \equiv \mathcal{A} + \mathcal{B}, \tag{1.4.10}$$

$$(\alpha a_{jk}) \equiv \alpha \mathcal{A}, \tag{1.4.11}$$

$$\left(\sum_{m=1}^{n} a_{jm} b_{mk}\right) \equiv \mathcal{A}\mathcal{B}. \tag{1.4.12}$$

A number of different but equivalent norms may be defined for $\mathfrak{M}_n$. A suitable one for analysis is

$$\|\mathcal{A}\| = \max_j \sum_{k=1}^{n} |a_{jk}|. \tag{1.4.13}$$

We have then

$$\|\mathcal{A}\mathcal{B}\| \leq \|\mathcal{A}\| \, \|\mathcal{B}\|.$$

Since $\mathfrak{M}_n$ is complete in the normed metric (why?), it follows that $\mathfrak{M}_n$ is a B-algebra.

We have seen that $\mathcal{A}$ has an inverse iff $\det(\mathcal{A}) \neq 0$. If this is the case, $\mathcal{A}$ is said to be *regular*; otherwise, *singular*. Together with the given matrix $\mathcal{A}$ we consider the family of matrices

$$\lambda \mathcal{E} - \mathcal{A},$$

where λ runs through the complex field $\mathbf{C}$ and $\mathcal{E} = (\delta_{jk})$ is the n-by-n unit matrix. These matrices are normally regular, but there exist n values of λ for which $\lambda \mathcal{E} - \mathcal{A}$ is singular: the n roots of the *characteristic equation* of $\mathcal{A}$,

$$\det(\lambda \mathcal{E} - \mathcal{A}) = 0. \tag{1.4.14}$$

The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ form the *spectrum* $\sigma(\mathcal{A})$ of $\mathcal{A}$. They are known as *characteristic values, latent roots,* or *eigenvalues.* For these values of λ one can find vectors $\mathbf{x}_k$ in $\mathbf{C}^n$ of norm 1 such that

$$\mathcal{A} \cdot \mathbf{x}_k = \lambda_k \mathbf{x}_k. \tag{1.4.15}$$

The *characteristic vectors* $\mathbf{x}_k$ are linearly independent and may be chosen so that they form an *orthogonal system*; in this case the *inner product*

$$(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{n} x_j \bar{y}_j = 0 \tag{1.4.16}$$

for $\mathbf{x} = \mathbf{x}_k$, $\mathbf{y} = \mathbf{x}_m$, $k \neq m$. This holds even if (1.4.14) has multiple roots. A matrix $\mathcal{A}$ is singular iff zero belongs to the spectrum.

EXERCISE 1.4

1. Find the elements of $\mathcal{A}^{-1}$ when $\mathcal{A}$ is regular.
2. Verify the inequality for the norm of the matrix product.
3. Prove the Hamilton-Cayley theorem, which asserts that the matrix $\mathcal{A}$ satisfies its own characteristic equation.

1.5. FIXED POINT THEOREMS

The Dutch mathematician L. E. J. Brouwer proved in 1912 that a continuous map of the unit ball in $\mathbb{R}^n$ into itself must necessarily leave at least one point invariant. Such a point is known as a *fixed point*, and an assertion about the existence of fixed points is known as a *fixed point theorem*. We shall prove some theorems of this nature. We start with a theorem proved by S. Banach in his Krakow dissertation of 1922. It refers to mappings of a complete metric space by a *contraction*, i.e., a bounded transformation of the space X into itself such that

$$d[T(x), T(y)] \leq k d(x, y), \tag{1.5.1}$$

where k is a fixed constant, $0 < k < 1$. Such a mapping evidently tries to shrink the object. Banach's theorem states that there is a point which does not move.

THEOREM 1.5.1

If T *is a contraction defined on a complete metric space* X, *then there is one and only one fixed point.*

Proof. The triangle inequality plays a basic role here. We start with an arbitrary point $x_1 \in X$ and form its successive transforms under T:

$$x_{n+1} = T(x_n), \qquad n = 1, 2, \ldots. \tag{1.5.2}$$

These elements form a Cauchy sequence, and X being a complete metric space, $x_0 = \lim x_n$ exists and is to be proved to be a fixed point—in fact, the only such point. Now it is sufficient to prove that, given any $\epsilon > 0$, there is an N such that

$$d(x_n, x_{n+p}) < \epsilon, \qquad n > N, \quad p = 1, 2, \ldots.$$