R. K. MELLUISH

An Introduction to the Mathematics of Map Projections

CAMBRIDGE

AN INTRODUCTION TO
THE MATHEMATICS OF
MAP PROJECTIONS

BY

R. K. MELLUISH, M.A.

Late Scholar of Gonville and Caius College, Cambridge. Assistant Master at Rossall School

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THE MATHEMATICS OF MAP PROJECTIONS

PREFACE

This book is the outcome of a mathematical essay on "Maps" written at Cambridge in 1922, and is an attempt to supply the need, which I then discovered, for a book dealing comprehensively with the theories that underlie their construction. It is therefore a book for a student of the mathematical side of geography; and a fair knowledge of the Calculus is all he will need to enable him to fill in for himself the many details of the work which have, for the sake of brevity, been omitted from the text.

I have attempted, in the earlier chapters, first of all to trace, as far as possible, the history of the projections concerned; this is followed by an account of the general theory, from which results are then deduced in the special cases that arise. In this way I deal with the properties of the four main classes of projections. Then follows a chapter on the theory of the Indicatrix, and the method of comparing one projection with another; next the question of finite measurements, perhaps the most important of all from a practical point of view; then a discussion of the best projection for a given country, and finally the general problem of conformal representation of which the complete solution has yet to be effected.

Though I have checked as well as I can the many expansions that are given, I am fully aware that some errors, I hope only slight, may have crept in; and I should therefore be pleased to receive corrections from any reader who may have discovered one.

I am indebted for many valuable suggestions to Mr. A. R. Hinks, F.R.S., Secretary of the Royal Geographical Society, who kindly lent me for reference the works by Germain and Tissot which appear in the list below. A great source of further help, especially in the work on "Minimum Error," was Mr A. E. Young's book, published as No. I of the Technical Series by the same Society.

The following is a complete list of the authors and works consulted, and I trust that, where necessary, adequate acknowledgment to them has been made in the text.

A. Germain. *Traité des Projections des Cartes Géographiques.* Paris, 1865.

A. Tissot. *Mémoire sur la Représentation des Surfaces et les Projections des Cartes Géographiques.* Paris, 1881.

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R. K. M

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Oct. 1931

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CHAPTER I

INTRODUCTORY

In all probability the drawing of maps and charts was closely associated in its beginnings with the study of geometry. According to Herodotus the Egyptians were brought to the study of geometry in the endeavour to keep records of the extent of their land, so that after the floods of the Nile it might be possible to assess the tax that each man had to pay according to the area of the land left to him. Thus they began to draw diagrams and charts of the divisions of the land, and these, no doubt, eventually grew into maps. Ignorance of the actual shape of the earth probably prevented them from realising many of the difficulties surrounding such a problem as the construction of a map, for ever since those days men have been studying the question and attempting to find the best method of solving it. For the representation of the earth, an oblate spheroid, on a plane, is a problem that admits of no absolutely correct solution. A spheroid is not a developable surface, such as a cone or a cylinder, and thus it is impossible to imagine a piece of paper wrapped round the earth, on which the shapes of countries and continents could be described exactly, and which could then be unwrapped into a plane map.

Any representation that we can make, any map that we can draw, must be incorrect in certain respects. It may be so in all, and be made so that each property of it approximates as nearly as possible to the corresponding one on the earth, or, as is more generally the case, it may be made so as to be correct in one or two respects, and not at all in the others; for example, so as to sacrifice correctness of shape to that of area, or that of azimuth to that of distance.

When a map is being drawn, each point on it is fixed according to some given law which expresses the co-ordinates of that point on the map in terms of those of the corresponding point on the earth. Such a law is called the Projection on which the map is drawn; and the equations of the projection are those which give the relation between the terrestrial co-ordinates and those of the point on the map. It is usually convenient to have the map co-ordinates expanded in ascending powers of the latitude and longitude, or differences of those quantities, of the point under consideration, and these expansions will of course be the solutions of the equations of the projection.

Projections in which the shape of small elements is preserved are called Orthomorphic, those in which areas are preserved Equal Area, and those which give distances of all points from a fixed centre correctly Simple or Equidistant. These are the three chief classes of projections, though there are several others which have been used to a less extent, for example, the Minimum Error, in which the total square error, i.e. the sum of the squares of the errors of scale in two directions at right angles, summed for every point of the map, is made a minimum—and others, such as the doubly azimuthal, of which some account is given later, which have as yet been put to no practical use.

Co-ordinates and length of arc of meridian.

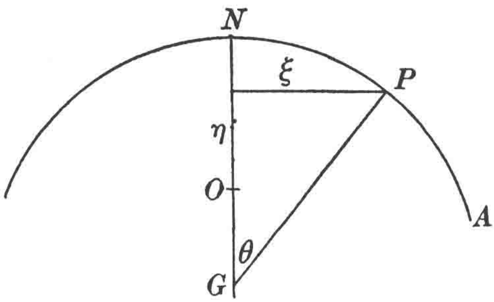
The position of a point on the earth is usually given by its latitude and longitude, but the mathematics is often simplified if, instead of the former of these, we make use of the colatitude, i.e. the angle between the normal at the point and the polar axis.

Let P be any point on the earth, PG the normal at P, and NOG the axis of the earth; and let $P\hat{G}N = \theta$, and the longitude of the meridian NPA be ϕ. Then if we take the earth as having an equatorial radius unity and eccentricity ϵ, the equation of the meridian NPA referred to its principal axes is

$$x^2 + \frac{y^2}{1-\epsilon^2} = 1.$$

It will sometimes be found convenient to use, instead of the eccentricity ϵ, the ellipticity, i.e. the ratio of the difference between the semi-axes to the semi-major axis. Calling this e we have

$$(1-e)^2 = 1 - \epsilon^2,$$



or, neglecting powers of e above the first, $e = \frac{\epsilon^2}{2}$. Thus the equation of the meridian is

$$x^2 + \frac{y^2}{1-2e} = 1.$$

If P be the point ξ, η the equation of the normal PG is

$$\frac{x-\xi}{\xi} = \frac{y-\eta}{\eta}(1-\epsilon^2),$$

whence $$\tan\theta = \frac{\xi(1-\epsilon^2)}{\eta}.$$

Also $$\xi^2 + \frac{\eta^2}{1-\epsilon^2} = 1,$$

$$\therefore \quad \xi = \frac{\sin\theta}{(1-\epsilon^2\cos^2\theta)^{\frac{1}{2}}} \quad \text{or} \quad \sin\theta\,(1 + e\cos^2\theta),$$

$$\eta = \frac{(1-\epsilon^2)\cos\theta}{(1-\epsilon^2\cos^2\theta)^{\frac{1}{2}}} \quad \text{or} \quad \cos\theta\,\{1 - e\,(2-\cos^2\theta)\}.$$

To find the element $d\sigma$ of arc of the meridian we have

$$\frac{d\sigma}{d\theta} = \left[\left(\frac{d\xi}{d\theta}\right)^2 + \left(\frac{d\eta}{d\theta}\right)^2\right]^{\frac{1}{2}} = \frac{1-\epsilon^2}{(1-\epsilon^2\cos^2\theta)^{\frac{3}{2}}},$$

or

$$1 - \frac{e}{2} + \frac{3e}{2}\cos 2\theta;$$

and the length of the arc of a meridian between two points of colatitudes α and β is to the first order

$$2\left(1-\frac{e}{2}\right)\delta + \frac{3e}{2}\cos 2\chi \sin 2\delta,$$

where

$$2\chi = \alpha + \beta, \quad 2\delta = \alpha - \beta.$$

Also if ρ and ν be the radius of curvature and normal of meridian respectively,

$$\rho = \frac{1-\epsilon^2}{(1-\epsilon^2\cos^2\theta)^{\frac{3}{2}}} \quad \text{or} \quad 1 - \frac{e}{2} + \frac{3e}{2}\cos 2\theta,$$

$$\nu = \xi \operatorname{cosec}\theta = \frac{1}{(1-\epsilon^2\cos^2\theta)^{\frac{1}{2}}} \quad \text{or} \quad 1 + e\cos^2\theta.$$

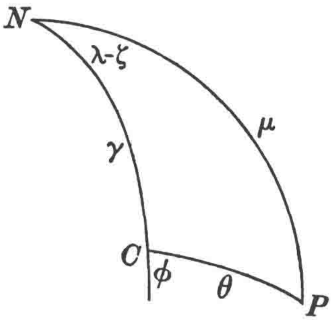
Normal, oblique and transverse projections.

If we neglect e altogether and regard the earth as a sphere, which is quite often sufficient for maps on small scales such as are used in atlases, then we simply have

$$\xi = \sin\theta, \quad \eta = \cos\theta.$$

In this case, since all the meridians are circles, it becomes possible to measure the latitude and longitude from an axis other than the geographical one. Thus out of one projection, giving the co-ordinates of the map point in terms of the geographical latitude and longitude of the earth point, we may derive another by regarding these latter as being no longer referred to the polar axis but to an axis through some other point. The original projection is called Normal, and the derived one Oblique, or, if the pole of the map be on the Equator, Transverse.

To obtain the relations between geographical co-ordinates and those referred to another pole, when regarding the earth as a sphere, we make use of the ordinary results of spherical trigonometry.



Let C be the pole of the map, N the north pole, and the co-latitude and longitude of C be γ, ζ. Then if P is any point whose co-ordinates referred to N are μ, λ and referred to C are θ, ϕ, we have—since, as we may measure ϕ from any plane, we may suppose it measured from that of the meridian through C—

$$\cos\theta = \cos\mu\cos\gamma + \sin\mu\sin\gamma\cos(\lambda - \zeta),$$

$$\cos\mu = \cos\theta\cos\gamma - \sin\theta\sin\gamma\cos\phi,$$

$$\sin\theta\sin\phi = \sin\mu\sin(\lambda - \zeta),$$

from which we may derive, by eliminating μ,

$$\cos\theta\sin\gamma + \sin\theta\cos\gamma\cos\phi = \sin\theta\sin\phi\cot(\lambda - \zeta).$$

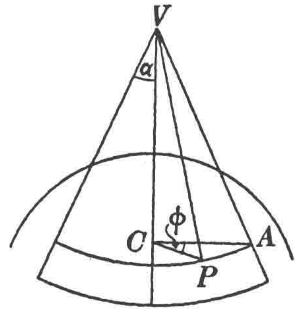
These equations enable us, when given the co-ordinates of a point on the map in terms of θ and ϕ, to find equations between them and their geographical co-ordinates μ and λ.

Conical projections.

A conical projection is one in which the plane map is derived from one drawn on a cone, simply unwrapping it from the cone. Imagine a cone, vertex V, placed so that its axis coincides with that of the earth, as in the figure, and suppose the projection be made in such a way that the azimuths of all points are preserved and the parallels on the earth become the circular cross-sections of the cone. Then we shall have a map on the cone in which the meridians on the earth are represented by the generators. Now if P be a point, longitude ϕ, and A the point on the meridian 0° and of the same latitude

$$A\hat{V}P = \frac{AP}{VP} = \frac{CP}{VP}\phi = \phi \sin \alpha,$$

where α is the semi-vertical angle of the cone. Thus if two points on the earth have their longitudes differing by ϕ, the angle between the generators through them is $\phi \sin \alpha$. Now let the map be unwrapped from the cone into a plane; we shall have for the parallels arcs of concentric circles, and for the meridians a set of concurrent straight lines making with each other angles proportional to the differences in longitude, and the constant of the proportion, which is called the constant of the cone, is $\sin \alpha$. Thus a map of the world would be enclosed between two lines inclined at an angle of $2\pi \sin \alpha$.



In the particular cases when the semi-vertical angle of the cone has its limiting values 0 and $\pi/2$, the cone becomes in the first case a cylinder and the projection is said to be cylindrical, in the second a plane and the projection is called Azimuthal, since the azimuths of all points from the pole are given correctly. A projection of this latter kind has also been called Zenithal, but there seems to be little reason for such a name.

A conical, cylindrical or azimuthal projection can of course be made to satisfy any other condition, for we have still to specify in what way the map on the cone is derived from the earth; for example it may be derived so that shapes are preserved, giving an orthomorphic, or areas, giving an equal area projection. Again the cone may be applied, if we regard the earth as a sphere, with its axis coinciding with some diameter other than the polar axis, and in this case we should have an oblique conical projection.

The equations eventually obtained by these considerations will contain certain constants, e.g. the constant of the cone, and we may determine these so as to satisfy certain other kinds of conditions. One method is to make the lengths of two parallels correct, that is, equal to the lengths of the corresponding parallels on the earth, subject to the modification of the scale on which the map is drawn. The cone of the fig. on p. 6, or the cylinder, in the case of a cylindrical projection, cuts the earth in these two parallels, which are said to be Standard Parallels. A modification of this method is to make the cone or cylinder touch the earth, in which case the two parallels coincide and the projection has one standard parallel only.

In the case of a conical projection with two standard parallels of colatitudes θ_1 and θ_2, and radii on the map r_1 and r_2, we shall have, taking the radius of the earth as unity—as we shall do throughout—

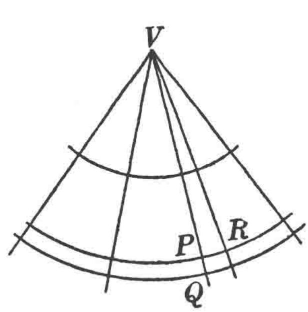
$$2\pi \sin \theta_1 = 2n\pi r_1,$$

and therefore $\sin \theta_1 = nr_1$ or $\xi_1 = nr_1$,

$$\sin \theta_2 = nr_2, \qquad \xi_2 = nr_2$$

for the spheroid. From these two equations n and the other constant which appears in the expression for the radius may be found.

Another method of calculating the constants, which has been used by Sir George Airy, Col. Clarke, and more latterly by Mr A. E. Young, is that of Minimum Error, a method practicable only in the case of conical projections. Suppose P any point on the map, co-ordinates θ, ϕ, and let the parallel through it have a radius r. Let Q be a neighbouring point on the same meridian, co-ordinates $\theta + \delta\theta$, ϕ; and R a neighbouring point on the same parallel,



co-ordinates θ, $\phi + \delta\phi$. Then the angle subtended at the centre V by PR is $n d\phi$, where n is the constant of the cone. We thus have $PQ = \delta r$, $PR = nr\,\delta\phi$; and the corresponding lengths on the earth are $\delta\theta$ and $\sin\theta\,\delta\phi$. Thus the scales along the meridian and the parallel are respectively

$$\frac{dr}{d\theta} \quad \text{and} \quad \frac{nr}{\sin\theta}.$$

Sir George Airy devised the plane of making the total square error, i.e. the sum of the squares of the errors of scale in these two directions, summed for every point of the map, a minimum. We should thus have to make

$$\iint \left[\left(\frac{dr}{d\theta} - 1 \right)^2 + \left(\frac{nr}{\sin\theta} - 1 \right)^2 \right] \sin\theta\, d\theta\, d\phi$$

a minimum, where the integration is taken for the whole area of the map, the factor $\sin\theta\, d\theta\, d\phi$ being the element of area on the earth, regarded as a sphere*. Airy only considered the case $n = 1$, i.e. an azimuthal projection, and found an expression for r in terms of θ, but Young, in his *Some Investigations in the Theory of Map Projections*, extended the method to the calculation of the constants of the projections as well as to the discovery of the equations.

A third method of calculating the constants, probably introduced first by Murdoch, is to make the total area of the map correct. As an example take the conical projection of the zone between two parallels α and β, the radii of whose projections are r_α and r_β. The area on the map is $(r_\beta{}^2 - r_\alpha{}^2)\, n\pi$, and that on the earth is

$$\int_\alpha^\beta \int_0^{2\pi} \sin\theta\, d\theta\, d\phi = 2\pi (\cos\alpha - \cos\beta).$$

* In a conical projection r is independent of ϕ, and if, as is usual, we are calculating the total square error of a zone between two parallels α and β, we can integrate with respect to ϕ at once, between the limits 0 and 2π, and dividing by 2π obtain the expression which we call the total square error

$$M = \int_\alpha^\beta \left[\left(\frac{dr}{d\theta} - 1 \right)^2 + \left(\frac{nr}{\sin\theta} - 1 \right)^2 \right] \sin\theta\, d\theta.$$