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Lars Hörmander

The Analysis of Linear Partial Differential Operators II

线性偏微分算子分析

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Differential Operators with Constant Coefficients

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Preface

This volume is an expanded version of Chapters III, IV, V and VII of my 1963 book "Linear partial differential operators". In addition there is an entirely new chapter on convolution equations, one on scattering theory, and one on methods from the theory of analytic functions of several complex variables. The latter is somewhat limited in scope though since it seems superfluous to duplicate the monographs by Ehrenpreis and by Palamodov on this subject.

The reader is assumed to be familiar with distribution theory as presented in Volume I. Most topics discussed here have in fact been encountered in Volume I in special cases, which should provide the necessary motivation and background for a more systematic and precise exposition. The main technical tool in this volume is the Fourier-Laplace transformation. More powerful methods for the study of operators with variable coefficients will be developed in Volume III. However, the constant coefficient theory has given the guidelines for all that work. Although the field is no longer very active - perhaps because of its advanced state of development - and although it is possible to pass directly from Volume I to Volume III, the material presented here should not be neglected by the serious student who wants to gain a balanced perspective of the theory of linear partial differential equations.

I would like to thank all who have helped me in various ways during the preparation of this volume. As in the case of the first Volume I am particularly indebted to Niels Jørgen Kokholm of the University of Copenhagen who has read all the proofs and in doing so suggested many improvements of the text.

Lund in February 1983 Lars Hörmander

Contents

Introduction

Differential operators with constant coefficients acting on distributions of compact support are diagonalized by the Fourier-Laplace transformation. Already in Chapter VII we showed how this observation leads to general results on existence of fundamental solutions and approximation of solutions. In Chapter X we first examine the regularity properties of the fundamental solutions more closely, including the location of the wave front set. Existence and approximation of solutions is then studied systematically. Hypoelliptic operators are characterized in Chapter XI; the typical examples of elliptic operators or the heat operator were discussed in Volume I. We also prove a general result on propagation of regularity of solutions which is similar to Holmgren's uniqueness theorem. In Volume I we gave explicit formulas for the solution of the Cauchy problem for the wave equation. The hyperbolic operators which have similar properties are investigated in Chapter XII which also includes a study of the characteristic Cauchy problem modelled on say the heat or Schrödinger equation. The precision of the results in these chapters pays off in Chapter XIII where it allows us to treat a fairly large class of operators with variable coefficients locally as perturbations of constant coefficient operators. The new features which appear even for such operators are emphasized by a discussion of non-uniqueness for the Cauchy problem. Chapter XIV is devoted to perturbation theory in $\mathbb{R}^n$ (short range scattering theory).

The study of general overdetermined systems of differential operators with constant coefficients requires more prerequisites from the theory of analytic functions of several complex variables than we wish to assume here. As mentioned in the preface several monographs have already been devoted to this topic. In Chapter XV we shall therefore only develop some of the basic analytic techniques in a way which simplifies the existing treatments. Their use is illustrated in the case of a single differential equation. As in the preceding chapters it would cause no additional difficulties to consider determined systems.

Chapter XVI is devoted to convolution equations. The translation invariance which they share with differential operators with constant coefficients makes their study so closely related that this enlargement of our main theme seems natural if not unavoidable.

Chapter X. Existence and Approximation of Solutions of Differential Equations

Summary

In the preceding chapters we have constructed a number of explicit fundamental solutions. In Chapter VII we also gave a construction which is applicable to any differential operator with constant coefficients. We return to it in Section 10.2 to discuss the regularity properties of the fundamental solutions obtained in greater detail. First we determine when a fundamental solution is in one of the spaces introduced in Section 10.1. These generalize the $H_{(s)}$ spaces of Section 7.9 and are defined essentially as inverse Fourier transforms of L^p spaces with respect to appropriate densities. In Section 10.2 we also examine how the fundamental solution $E(P)$ of $P(D)$ given by the construction depends on P, and we estimate $WF(E(P))$ for a fixed P.

A differential equation $P(D)u=f$ with $f\in\mathscr{E}'$ can immediately be solved if one has a fundamental solution of $P(D)$ available. In Section 10.3 we determine the properties of the solution rather precisely by means of the results on fundamental solutions proved in Section 10.2. This leads to a comparison of the relative strengths of differential operators (polynomials); P is said to be stronger than Q if $Q(D)u$ is at least as regular as $P(D)u$ when $u\in\mathscr{E}'$. This notion, in a more precise form, is studied at some length in Section 10.4. In Section 10.5 we give a brief discussion of approximation theorems of Runge's type for solutions of the homogeneous differential equation $P(D)u=0$. This prepares for the study in Section 10.6 of the differential equation $P(D)u=f$ when f is a distribution of finite order. The same problem is discussed in Section 10.7 when f is an arbitrary distribution. Finally, Section 10.8 is devoted to the search for a geometrical form of the conditions for solvability encountered in Sections 10.6 and 10.7.

10.1. The Spaces $B_{p,k}$

In an existence theory for partial differential equations it is important to give precise statements concerning the regularity of the solutions obtained. Now a condition on the regularity of a distribution or function u (with compact support) can also be regarded as a condition on the behavior of the Fourier transform $\hat{u}$ at infinity. To classify this behavior one may for example ask for which weight functions k it is true that $k\hat{u}\in L^p$. The set of all such temperate distributions u is denoted by $B_{p,k}$ here. Only the cases $p=2$, $p=\infty$ and $p=1$ are really interesting. Concerning k we shall make a hypothesis which ensures that $B_{p,k}$ is a module over C_0^∞:

Definition 10.1.1. A positive function k defined in $\mathbb{R}^n$ will be called a temperate weight function if there exist positive constants C and N such that

$$k(\xi+\eta)\leqq(1+C|\xi|)^N k(\eta); \quad \xi,\eta\in\mathbb{R}^n. \tag{10.1.1}$$

The set of all such functions k will be denoted by $\mathscr{K}$.

Remark. In harmonic analysis a weight function usually means a function K such that

$$K(\xi+\eta)\leqq K(\xi)\,K(\eta).$$

To avoid confusion we shall call such functions submultiplicative here.

From (10.1.1) it follows that

$$(1+C|\xi|)^{-N}\leqq k(\xi+\eta)/k(\eta)\leqq(1+C|\xi|)^N; \quad \xi,\eta\in\mathbb{R}^n. \tag{10.1.1$'$}$$

In fact, the left-hand inequality is obtained if η is replaced by $\xi+\eta$ and ξ is replaced by $-\xi$ in (10.1.1). If we let $\xi\to 0$ in (10.1.1)$'$ it follows that k is continuous, and when $\eta=0$ we obtain the useful estimates

$$k(0)(1+C|\xi|)^{-N}\leqq k(\xi)\leqq k(0)(1+C|\xi|)^N. \tag{10.1.2}$$

If $k\in\mathscr{K}$ we shall write

$$M_k(\xi)=\sup_\eta k(\xi+\eta)/k(\eta). \tag{10.1.3}$$

This means that M_k is the smallest function such that

$$k(\xi+\eta)\leqq M_k(\xi)\,k(\eta). \tag{10.1.4}$$

It is clear that M_k is submultiplicative,

$$M_k(\xi+\eta)\leqq M_k(\xi)\,M_k(\eta), \tag{10.1.5}$$

and since $M_k(\xi)\leqq(1+C|\xi|)^N$ this implies that $M_k\in\mathscr{K}$. From (10.1.5) we can deduce that

$$1=M_k(0)\leqq M_k(\xi), \quad \xi\in\mathbb{R}^n. \tag{10.1.6}$$

In fact, for every positive integer ν we have

$$1=M_k(0)\leqq M_k(\xi)^\nu M_k(-\nu\xi)\leqq M_k(\xi)^\nu(1+C\nu|\xi|)^N,$$

and if we take ν^{th} roots in this inequality and let $\nu\to\infty$, the estimate (10.1.6) follows.

Example 10.1.2. The spaces $H_{(s)}$ in Section 7.9 correspond to

$$k_s(\xi)=(1+|\xi|^2)^{s/2}.$$

That $k_s\in\mathscr{K}$ and that $M_{k_s}(\eta)\leqq(1+|\eta|)^{|s|}$ follows from the estimates

$$1+|\xi+\eta|^2\leqq 1+|\xi|^2+2|\xi||\eta|+|\eta|^2\leqq(1+|\xi|^2)(1+|\eta|)^2.$$

Example 10.1.3. The basic example of a function in $\mathscr{K}$, which is the reason why we introduce it here, is the function $\tilde{P}$ defined by

$$\tilde{P}(\xi)^2=\sum_{|\alpha|\geqq 0}|P^{(\alpha)}(\xi)|^2 \tag{10.1.7}$$

where P is a polynomial so that the sum is finite. Here $P^{(\alpha)}=\partial^\alpha P$. It follows immediately from Taylor's formula that

$$\tilde{P}(\xi+\eta)\leqq(1+C|\xi|)^m\tilde{P}(\eta), \tag{10.1.8}$$

where m is the degree of P and C a constant depending only on m and the dimension n. From this example other functions in $\mathscr{K}$ are obtained in the theory as a result of the operations described in the next theorem.

Theorem 10.1.4. *If k_1 and k_2 belong to $\mathscr{K}$, it follows that k_1+k_2, k_1k_2, $\sup(k_1,k_2)$ and $\inf(k_1,k_2)$ are also in $\mathscr{K}$. If $k\in\mathscr{K}$ we have $k^s\in\mathscr{K}$ for every real s, and if μ is a positive measure we have either $\mu*k\equiv\infty$ or else $\mu*k\in\mathscr{K}$.*

Proof. It follows from (10.1.1)′ that $1/k\in\mathscr{K}$ if $k\in\mathscr{K}$. The statements are thus all trivial except perhaps the last. To prove that one, we note that (10.1.1) gives

$$(\mu*k)(\xi+\eta)\leqq(1+C|\xi|)^N(\mu*k)(\eta).$$

If $\mu*k$ is finite for some η it follows that $\mu*k$ is finite everywhere and belongs to $\mathscr{K}$.

Occasionally it is useful to know that functions in $\mathscr{K}$ can be replaced by equivalent functions which vary very slowly indeed:

Theorem 10.1.5. *If $k\in\mathscr{K}$ we can for every $\delta>0$ find a function $k_\delta\in\mathscr{K}$ and a constant C_δ such that*

(10.1.9) $$1 \leqq k_\delta(\xi)/k(\xi) \leqq C_\delta, \qquad \xi \in \mathbb{R}^n,$$

(10.1.10) $$M_{k_\delta}(\xi) \leqq (1+C|\xi|)^N, \qquad \xi \in \mathbb{R}^n,$$

where C *and* N *are independent of* δ, *and*

(10.1.11) $M_{k_\delta} \to 1$ *uniformly on compact subsets of* $\mathbb{R}^n$ *when* $\delta \to 0$.

Proof. We shall set

$$k_\delta(\xi) = \sup_\eta e^{-\delta|\eta|} k(\xi-\eta).$$

(Note the analogy with the definition of a convolution.) Then we have in view of (10.1.1)

$$k(\xi) \leqq k_\delta(\xi) \leqq \sup_\eta e^{-\delta|\eta|}(1+C|\eta|)^N k(\xi) = C_\delta k(\xi)$$

where the last equality is a definition of C_δ. This proves (10.1.9). To prove (10.1.10) we note that

$$\begin{aligned} k_\delta(\xi+\xi') &= \sup_\eta e^{-\delta|\eta|} k(\xi+\xi'-\eta) \\ &\leqq \sup_\eta e^{-\delta|\eta|}(1+C|\xi'|)^N k(\xi-\eta) \\ &= (1+C|\xi'|)^N k_\delta(\xi); \qquad \xi, \xi' \in \mathbb{R}^n. \end{aligned}$$

To prove (10.1.11) we first rewrite the definition of k_δ by introducing $\xi-\eta$ as a variable instead of η. This gives

$$k_\delta(\xi) = \sup_\eta e^{-\delta|\xi-\eta|} k(\eta).$$

Hence

$$\begin{aligned} k_\delta(\xi+\xi') &= \sup_\eta e^{-\delta|\xi+\xi'-\eta|} k(\eta) \\ &\leqq e^{\delta|\xi'|} \sup_\eta e^{-\delta|\xi-\eta|} k(\eta) = e^{\delta|\xi'|} k_\delta(\xi), \end{aligned}$$

which proves that

$$1 \leqq M_{k_\delta}(\xi) \leqq e^{\delta|\xi|}$$

The proof is complete.

Remark. It might have been more natural to require in Definition 10.1.1 only that k is continuous and that

$$k(\xi+\eta) \leqq C(1+|\xi|)^N k(\eta).$$

We have not done so since this would not guarantee the continuity of M_k. Now the proof of Theorem 10.1.5 shows that for such functions we still obtain (10.1.9) and $k_\delta \in \mathscr{K}$. Our choice of definition has therefore not led to any significant loss of generality.

We can now give the formal definition of the spaces we need:

Definition 10.1.6. If $k \in \mathcal{K}$ and $1 \leqq p \leqq \infty$, we denote by $B_{p,k}$ the set of all distributions $u \in \mathcal{S}'$ such that $\hat{u}$ is a function and

$$\|u\|_{p,k} = ((2\pi)^{-n} \int |k(\xi)\, \hat{u}(\xi)|^p d\xi)^{1/p} < \infty. \tag{10.1.12}$$

When $p=\infty$ we shall interpret $\|u\|_{p,k}$ as ess. $\sup|k(\xi)\, \hat{u}(\xi)|$.

The factor $(2\pi)^{-n}$ is included for convenience in the results and is motivated by the fact that the measure $(2\pi)^{-n} d\xi$ occurs in Parseval's formula. For example, we have thanks to this normalization

$$\|u\|_{2,\tilde{P}} = (\sum_\alpha \|P^{(\alpha)}(D)\, u\|_{L^2}^2)^{\frac{1}{2}}.$$

Theorem 10.1.7. *$B_{p,k}$ is a Banach space with the norm* (10.1.12). *We have*

$$\mathcal{S} \subset B_{p,k} \subset \mathcal{S}',$$

also in the topological sense.[1] *C_0^∞ is dense in $B_{p,k}$ if $p < \infty$.*

Proof. Let $L_{p,k}$ be the Banach space of all measurable functions v such that the norm $(2\pi)^{-n/p} \|kv\|_p < \infty$. Then we have

$$\mathcal{S} \subset L_{p,k} \subset \mathcal{S}',$$

also in the topological sense, and $\mathcal{S}$ is dense in $L_{p,k}$ if $p < \infty$. In fact, that $\mathcal{S} \subset L_{p,k}$ follows from the second inequality in (10.1.2), and if $p < \infty$ it follows from Theorem 1.3.2 that even C_0^∞ is dense in $L_{p,k}$, for C_0^0 is dense there. To prove that $L_{p,k} \subset \mathcal{S}'$ we note that Hölder's inequality gives

$$\int |\phi v|\, d\xi \leqq \|kv\|_p \|\phi/k\|_{p'}$$

where $1/p + 1/p' = 1$. This proves our assertion since $\|\phi/k\|_{p'}$ is a continuous semi-norm in $\mathcal{S}$ in view of the first inequality in (10.1.2). If we now use the fact that the Fourier transformation is an automorphism of $\mathcal{S}$ and of $\mathcal{S}'$, it follows that $B_{p,k}$ is complete, that $\mathcal{S} \subset B_{p,k} \subset \mathcal{S}'$ (topologically) and that $\mathcal{S}$ is dense in $B_{p,k}$ if $p < \infty$. Since C_0^∞ is dense in $\mathcal{S}$ by Lemma 7.1.8, this completes the proof.

Theorem 10.1.8. *If $k_1, k_2 \in \mathcal{K}$ and*

$$k_2(\xi) \leqq C k_1(\xi), \quad \xi \in \mathbb{R}^n, \tag{10.1.13}$$

it follows that $B_{p,k_1} \subset B_{p,k_2}$. Conversely, if there exists an open set $X \neq \emptyset$ such that $B_{p,k_1} \cap \mathcal{E}'(X) \subset B_{p,k_2}$, then (10.1.13) *is valid.*

Proof. The first part of the theorem is trivial. To prove the second we let F be a compact subset of X with non-empty interior, and set

[1] This means that the topology in $\mathcal{S}$ is stronger than that induced there by $B_{p,k}$ and that the topology in $B_{p,k}$ is stronger than the one induced by $\mathcal{S}'$.

$B = B_{p,k_1} \cap \mathscr{E}'(F)$. Since the topology in each of the spaces B_{p,k_j} is stronger than that in $\mathscr{S}'$ (Theorem 10.1.7), it is clear that B is a closed subspace of B_{p,k_1} and that the inclusion mapping of B into B_{p,k_2} is closed. Hence it follows from the closed graph theorem that

$$\|u\|_{p,k_2} \leqq C_1 \|u\|_{p,k_1}, \quad u \in B, \tag{10.1.14}$$

where C_1 is a constant. With a fixed function $u \in C_0^\infty(F)$ such that $u \not\equiv 0$, we shall apply (10.1.14) to $u_\eta(x) = u(x) e^{i\langle x,\eta\rangle}$ where $\eta \in \mathbb{R}^n$. Since $\hat{u}_\eta(\xi) = \hat{u}(\xi - \eta)$, the estimates

$$|k_1(\xi)\, \hat{u}(\xi-\eta)| \leqq k_1(\eta) |M_{k_1}(\xi-\eta)\, \hat{u}(\xi-\eta)|,$$
$$|k_2(\xi)\, \hat{u}(\xi-\eta)| \geqq k_2(\eta) |\hat{u}(\xi-\eta)/M_{k_2}(\eta-\xi)|$$

give

$$\|u_\eta\|_{p,k_1} \leqq k_1(\eta) \|u\|_{p,M_{k_1}}; \quad \|u_\eta\|_{p,k_2} \geqq k_2(\eta) \|u\|_{p,1/\check{M}_{k_2}}. \tag{10.1.15}$$

If we combine these inequalities with (10.1.14) we obtain (10.1.13) with the constant $C = C_1 \|u\|_{p,M_{k_1}} / \|u\|_{p,1/\check{M}_{k_2}}$. The proof is complete.

Corollary 10.1.9. *If $k_1, k_2 \in \mathscr{K}$, it follows that*

$$B_{p,k_1} \cap B_{p,k_2} = B_{p,k_1+k_2}$$

and that

$$\max_{j=1,2} \|u\|_{p,k_j} \leqq \|u\|_{p,k_1+k_2} \leqq \|u\|_{p,k_1} + \|u\|_{p,k_2}, \quad u \in B_{p,k_1} \cap B_{p,k_2}.$$

Proof. Since $k_j \leqq k_1 + k_2$ we have

$$B_{p,k_1+k_2} \subset B_{p,k_j} \quad \text{and} \quad \|u\|_{p,k_j} \leqq \|u\|_{p,k_1+k_2}$$

for $j=1,2$. On the other hand, if $u \in B_{p,k_1} \cap B_{p,k_2}$, it follows from Minkowski's inequality that $u \in B_{p,k_1+k_2}$ and that the second part of the inequality is valid. This proves the corollary.

We next examine when the inclusion mapping in Theorem 10.1.8 is compact.

Theorem 10.1.10. *If K is a compact set in $\mathbb{R}^n$, the inclusion mapping of $B_{p,k_1} \cap \mathscr{E}'(K)$ into B_{p,k_2} is compact if*

$$k_2(\xi)/k_1(\xi) \to 0, \quad \xi \to \infty. \tag{10.1.16}$$

Conversely, if the mapping is compact for one set K with interior points, it follows that (10.1.16) *is valid.*

Proof. a) Assuming that (10.1.16) is valid, we take a sequence $u_\nu \in B_{p,k_1} \cap \mathscr{E}'(K)$ such that $\|u_\nu\|_{p,k_1} \leqq 1$. We have to prove that there exists a subsequence converging in B_{p,k_2}. Let ϕ be a function in $C_0^\infty(\mathbb{R}^n)$ which equals 1 in a neighborhood of K. Since $u_\nu = \phi u_\nu$, we have

$$\hat{u}_\nu(\xi) = (2\pi)^{-n} \int \hat{\phi}(\xi - \eta)\, \hat{u}_\nu(\eta)\, d\eta. \tag{10.1.17}$$

Multiplying (10.1.17) by $k_1(\xi)$ and using the inequality

$$k_1(\xi) \leqq M_{k_1}(\xi - \eta)\, k_1(\eta)$$

and Hölder's inequality, we obtain

$$\begin{aligned} |k_1(\xi)\, \hat{u}_\nu(\xi)| &\leqq (2\pi)^{-n} \|M_{k_1} \hat{\phi}\|_{p'} \|k_1 \hat{u}_\nu\|_p \\ &\leqq (2\pi)^{-n/p'} \|M_{k_1} \hat{\phi}\|_{p'}. \end{aligned}$$

Similarly, application of the same argument after differentiation of (10.1.17) gives

$$|k_1(\xi)\, D^\alpha \hat{u}_\nu(\xi)| \leqq (2\pi)^{-n/p'} \|M_{k_1} D^\alpha \hat{\phi}\|_{p'}.$$

This proves that the sequence $\hat{u}_\nu$ is uniformly bounded and equicontinuous on every compact set. Thus we can find a subsequence converging uniformly on all compact sets; changing notations, if necessary, we may assume that the sequence $\hat{u}_\nu$ itself is uniformly convergent on compact sets, Given any $\varepsilon > 0$ we now choose a ball S so large that $k_2(\xi)/k_1(\xi) < \varepsilon$ when $\xi \notin S$. Using Minkowski's inequality we obtain

$$\|u_\mu - u_\nu\|_{p,k_2} \leqq \varepsilon \|u_\mu - u_\nu\|_{p,k_1} + ((2\pi)^{-n} \int_S |k_2(\hat{u}_\mu - \hat{u}_\nu)|^p\, d\xi)^{1/p},$$

with the usual interpretation when $p = \infty$. The second term on the right-hand side tends to 0 when μ and $\nu \to \infty$, and the first is always $\leqq 2\varepsilon$. Hence the sequence u_ν is a Cauchy sequence in B_{p,k_2}, which proves the compactness.

b) Assuming the compactness of the inclusion mapping for some compact set K with interior points, we shall prove (10.1.16). To do so it is sufficient to prove that if a sequence $\xi_\nu \to \infty$ then $k_2(\xi_\nu)/k_1(\xi_\nu) \to 0$. Let $C_0^\infty(K) \ni u \not\equiv 0$, and set

$$u_\nu(x) = u(x)\, e^{i\langle x, \xi_\nu \rangle} / k_1(\xi_\nu).$$

From (10.1.15) we then obtain

$$\begin{aligned} &\|u_\nu\|_{p,k_1} \leqq \|u\|_{p,M_{k_1}}, \\ &\|u_\nu\|_{p,k_2} \geqq \|u\|_{p,1/\check{M}_{k_2}}\, k_2(\xi_\nu)/k_1(\xi_\nu). \end{aligned} \tag{10.1.18}$$

From the first of these inequalities it follows that the sequence u_ν is bounded in B_{p,k_1}, hence precompact in B_{p,k_2}. Now $u_\nu \to 0$ in $\mathscr{S}'$ for if