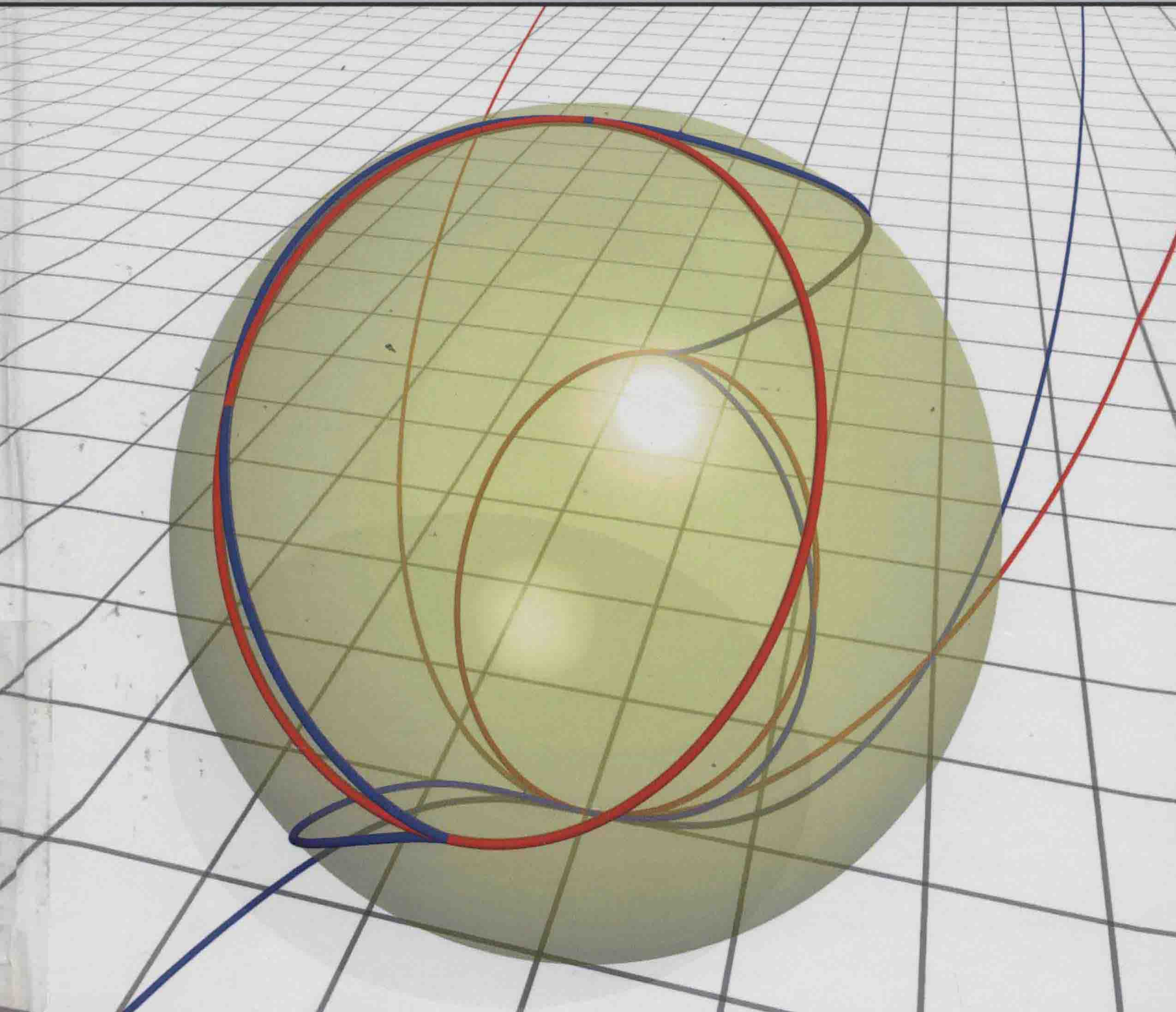
Advanced Concepts and Applications of Function Spaces

Denver Sosa



Advanced Concepts and Applications of Function Spaces

Edited by **Denver Sosa**

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Preface

Function space is a field of mathematics that studies some set of functions of some specific kind from one set X to another set Y, it defines the function, and the space stands for applications in which it can be both vector space or topological space or one of them at times. Functional space can be seen in many topics of mathematics, they play a vital role in problem solving in the topics like; set theory, linear algebra, linear transformation, functional analysis, topology, Homotopy theory, algebraic topology, stochastic processes' theory, map evaluation, calculus, etc.

Function spaces are very common theoretical concepts in the topics such as metric and normed spaces, Cauchy and convergent sequences, uniform limits of continuous functions, contraction mapping theorem, implicit function theorem, cauchy-schwarz inequality and parallelogram law, Bessel's inequality, fourier analysis. These are some of the very important topics that include function spaces in the foundation of applications. This book contains some practical as well as conceptual concepts of function spaces and the application description is also theorized in the contents.

I especially wish to acknowledge the contributing authors, without whom a work of this magnitude would clearly not be realizable. I thank them for allocating much of their very scarce time to this project. Not only do I appreciate their participation but also their adherence as a group to the time parameters set for this publication. I also wish to thank my publisher who considered me worthy of this incredible opportunity and supported me at every step.

Editor

Contents

Boundedness of n-Multiple Discrete Hardy Operators with Weights for $1 < q < p < \infty$

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We find necessary and sufficient conditions on weighted sequences ω_i, $i = 1, 2, \ldots, n-1$, u, and v, for which the operator $(S_{n-1}f)_i = \sum_{k_1=1}^{i} \omega_{1,k_1} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} \omega_{n-1,k_{n-1}} \sum_{j=1}^{k_{n-1}} f_j$, $i \geq 1$, is bounded from $l_{p,v}$ to $l_{q,u}$ for $1 < q < p < \infty$.

1. Introduction

Let $1 < p,\ q < \infty$ and $1/p + 1/p' = 1$. Let $v = \{v_k\}_{k=1}^{\infty}$ be positive, $\omega_i = \{\omega_{i,k}\}_{k=1}^{\infty}$, $i = 1, 2, \ldots, n-1$, and $u = \{u_k\}_{k=1}^{\infty}$ nonnegative number sequences, hereinafter referred to as weights.

Let $l_{p,v}$ be the space of sequences $f = \{f_k\}_{k=1}^{\infty}$, for which the following norm is finite:

$$\|f\|_{p,v} = \left(\sum_{i=1}^{\infty} |v_i f_i|^p \right)^{1/p}. \quad (1)$$

Consider the operator S_{n-1} that acts on the sequence $f = \{f_k\}_{k=1}^{\infty}$ as follows:

$$(S_{n-1}f)_i = \sum_{k_1=1}^{i} \omega_{1,k_1} \sum_{k_2=1}^{k_1} \omega_{2,k_2} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} \omega_{n-1,k_{n-1}} \sum_{j=1}^{k_{n-1}} f_j, \quad i \geq 1, \quad (2)$$

and call it n-multiple discrete Hardy operator with weights.

By changing the order of summation in (2) we have

$$(S_{n-1}f)_i = \sum_{j=1}^{i} A_{n-1,1}(i,j) f_j, \quad i \geq 1, \quad (3)$$

where $A_{n-1,1}(i,j) \equiv 1$ for $n = 1$ and

$$A_{n-1,1}(i,j) = \sum_{k_{n-1}=j}^{i} \omega_{n-1,k_{n-1}} \sum_{k_{n-2}=k_{n-1}}^{i} \omega_{n-2,k_{n-2}} \cdots \sum_{k_1=k_2}^{i} \omega_{1,k_1}, \quad i \geq j \geq 1, \quad (4)$$

for $n > 1$.

Together with operator (3) we consider its dual operator

$$(S^*_{n-1}f)_j = \sum_{i=j}^{\infty} A_{n-1,1}(i,j) f_i, \quad j \geq 1. \quad (5)$$

When $n = 1$, the operators S_{n-1} and S^*_{n-1} are simple discrete Hardy operators $(S_0 f)_i = \sum_{j=1}^{i} f_j$ and $(S_0^* f)_i = \sum_{j=i}^{\infty} f_j$; the problem of boundedness from $l_{p,v}$ to $l_{q,u}$ is investigated in detail in [1–3] for all values of the parameters p and q.

In paper [4] necessary and sufficient conditions of boundedness of operators (3) and (5) from $l_{p,v}$ to $l_{q,u}$ are obtained for the case $1 < p \leq q < \infty$. However, the problem of boundedness of these operators has not been studied for the case $1 < q < p < \infty$. In this paper we consider this problem.

Let us notice that the boundedness problem of continuous Hardy-type operators has been studied and developed in an extraordinary depth (see, e.g., [3]). The corresponding results for n-multiple integral Hardy operators were considered by Baiarystanov in [5]. Namely, he investigated the continuous analogue of operators (3) and (5) and found necessary and sufficient conditions of their boundedness from L_p to L_q for the case $1 < p \le q < \infty$. Moreover, there he obtained sufficient conditions of the same problem for the case $1 < q < p < \infty$. Later in [6] Sagindykov proved that sufficient conditions found by Baiarystanov for the case $1 < q < p < \infty$ are also necessary. However, the method for the continuous case does not work in the discrete situation as it was mentioned in [4]. Thus, here we present a completely different method.

In the sequel we suppose that the sum $\sum_{i=t}^{k}$ is equal to zero for $t > k$; the symbol $F \ll E$ means $F \le cE$, where a positive constant c does not depend on arguments of the expressions F and E but can depend on the parameters p and q. The relationship $F \approx E$ means $F \ll E \ll F$.

2. Auxiliary Statements

For all $i \ge j \ge 1$ we assume that $A_{l,m}(i,j) \equiv 1$ if $l < m$ and $A_{l,m}(i,j) = \sum_{k_l=j}^{i} \omega_{l,k_l} \sum_{k_{l-1}=k_l}^{i} \omega_{l-1,k_{l-1}} \cdots \sum_{k_m=k_{m+1}}^{i} \omega_{m,k_m}$ if $n-1 \ge l \ge m \ge 1$. Moreover, for all $i < j$ we suppose that $A_{l,m}(i,j) = 0$ if $l, m \ge 1$.

In paper [4] the following lemma is proved.

Lemma 1 (see [4]). *For all $i, j, \tau : 1 \le j \le \tau \le i$ one has*

$$\max_{m-1\le r\le l} A_{r,m}(i,\tau) A_{l,r+1}(\tau,j) \le A_{l,m}(i,j) \le \sum_{r=m-1}^{l} A_{r,m}(i,\tau) A_{l,r+1}(\tau,j), \qquad (6)$$

when $n-1 \ge l \ge m \ge 1$.

Lemma 2. *Let $(B_{i,j})_{i,j=1}^{\infty}$ be a nonnegative matrix whose elements do not decrease in the first index and do not increase in the second index. Let $\gamma > 0$. Then for $1 \le t,\ k \le \infty$ one has*

$$\Delta_j^{+}\left(\sum_{i=j}^{k} B_{i,j}\right)^{\gamma} \approx \left(\sum_{i=j}^{k} B_{i,j}\right)^{\gamma-1} \Delta_j^{+}\left(\sum_{i=j}^{k} B_{i,j}\right), \qquad (7)$$

$$\Delta_j^{-}\left(\sum_{j=t}^{i} B_{i,j}\right)^{\gamma} \approx \left(\sum_{j=t}^{i} B_{i,j}\right)^{\gamma-1} \Delta_i^{-}\left(\sum_{j=t}^{i} B_{i,j}\right), \qquad (8)$$

where $\Delta_j^{+} E_{i,j} = E_{i,j} - E_{i,j+1}$ and $\Delta_i^{-} E_{i,j} = E_{i,j} - E_{i-1,j}$.

Generally speaking the statement of Lemma 2 is known, but for more complete presentation let us give its proof. We prove only relation (7). Relation (8) can be proved similarly.

Proof. Let $\gamma \ge 1$. Then

$$\begin{aligned} \Delta_j^{+}\left(\sum_{i=j}^{k} B_{i,j}\right)^{\gamma} &= \left(\sum_{i=j}^{k} B_{i,j}\right)^{\gamma} - \left(\sum_{i=j+1}^{k} B_{i,j+1}\right)^{\gamma} \\ &\le \gamma \left(\sum_{i=j}^{k} B_{i,j}\right)^{\gamma-1} \Delta_j^{+}\left(\sum_{i=j}^{k} B_{i,j}\right), \\ \Delta_j^{+}\left(\sum_{i=j}^{k} B_{i,j}\right)^{\gamma} &\ge \left(\sum_{i=j}^{k} B_{i,j}\right)^{\gamma} - \left(\sum_{i=j}^{k} B_{i,j}\right)^{\gamma-1} \sum_{i=j+1}^{k} B_{i,j+1} \\ &= \left(\sum_{i=j}^{k} B_{i,j}\right)^{\gamma-1} \Delta_j^{+}\left(\sum_{i=j}^{k} B_{i,j}\right). \end{aligned} \qquad (9)$$

If $0 < \gamma < 1$, then these inequalities hold in opposite direction. Therefore (7) holds for all $\gamma > 0$. The proof of Lemma 2 is complete. □

Moreover, we need the following obvious relations:

$$\begin{aligned} \left(\sum_{i=t}^{k} B_{i,t}\right)^{\gamma} &= \sum_{j=t}^{k} \Delta_j^{+}\left(\sum_{i=j}^{k} B_{i,j}\right)^{\gamma}, \\ \left(\sum_{j=t}^{k} B_{k,j}\right)^{\gamma} &= \sum_{i=t}^{k} \Delta_i^{-}\left(\sum_{j=t}^{i} B_{i,j}\right)^{\gamma}. \end{aligned} \qquad (10)$$

We also use the following lemma.

Lemma 3 (see [1]). *Let $1 < p < q < \infty$. Then the operator S_0^{*} is bounded from $l_{p,v}$ to $l_{q,u}$ if and only if*

$$B_0(1) = \left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{i} u_j^{q}\right)^{p/(p-q)} \left(\sum_{k=i}^{\infty} v_k^{-p'}\right)^{p(q-1)/(p-q)} v_i^{-p'}\right)^{(p-q)/pq} < \infty; \qquad (11)$$

moreover, $B_0(1) \approx \| S_0^{*} \|$.

3. Main and Associated Results

For $r = 0, 1, \ldots, n-1$, we assume that

$$A_r(n) = \left(\sum_{i=1}^{\infty}\left(\sum_{j=i}^{\infty} A_{r,1}^{q}(j,i)\, u_j^{q}\right)^{q/(p-q)}\right.$$

$$\times\left(\sum_{k=1}^{i}A_{n-1,r+1}^{p'}(i,k)\,v_k^{-p'}\right)^{q(p-1)/(p-q)}$$

$$\times\Delta_i^+\left(\sum_{j=i}^{\infty}A_{r,1}^{q}(j,i)\,u_j^q\right)\Bigg)^{(p-q)/pq},$$

$$B_r(n)=\Bigg(\sum_{i=1}^{\infty}\left(\sum_{j=i}^{\infty}A_{r,1}^{p'}(j,i)\,v_j^{-p'}\right)^{p(q-1)/(p-q)}$$

$$\times\left(\sum_{k=1}^{i}A_{n-1,r+1}^{q}(i,k)\,u_k^q\right)^{p/(p-q)}$$

$$\times\Delta_i^+\left(\sum_{j=i}^{\infty}A_{r,1}^{p'}(j,i)\,v_j^{-p'}\right)\Bigg)^{(p-q)/pq}. \tag{12}$$

Theorem 4. *Let $1<q<p<\infty$ and $n\geq 1$. Then operator (3) is bounded from $l_{p,v}$ to $l_{q,u}$ if and only if $A(n)=\max_{0\leq r\leq n-1}A_r(n)<\infty$. Moreover, for the norm of operator (3) from $l_{p,v}$ to $l_{q,u}$ the following relation $\| S_{n-1}\|\approx A(n)$ holds.*

Theorem 5. *Let $1<q<p<\infty$ and $n\geq 1$. Then operator (5) is bounded from $l_{p,v}$ to $l_{q,u}$ if and only if $B(n)=\max_{0\leq r\leq n-1}B_r(n)<\infty$. Moreover, for the norm of operator (5) from $l_{p,v}$ to $l_{q,u}$ the following relation $\| S_{n-1}^*\|\approx B(n)$ holds.*

For the proofs of Theorems 4 and 5 we need to establish several statements.

Assume that

$$\widetilde{u}_i^q=\Delta_i^-\left(\sum_{j=1}^{i}A_{n-1,r+1}^{q}(i,j)\,u_j^q\right),$$

$$(S_rf)_i=\sum_{j=1}^{i}A_{r,1}(i,j)\,f_j, \tag{13}$$

$$(S_r^*g)_j=\sum_{i=j}^{\infty}A_{r,1}(i,j)\,g_i,\quad i,j\geq 1,\ 0\leq r\leq n-1.$$

Consider the inequality

$$\left\|S_r^*g\right\|_{q,\widetilde{u}}\leq C\left\|g\right\|_{p,v}. \tag{14}$$

Lemma 6. *Let $1<p,\ q<\infty$. Then the inequality*

$$\left\|S_rf\widetilde{u}^q\right\|_{p',v^{-1}}\leq C\left\|f\right\|_{q',\widetilde{u}^{q-1}},\quad f\geq 0, \tag{15}$$

is dual to (14) with respect to the linear form $\sum_{i=1}^{\infty}f_ig_i\widetilde{u}_i^q$.

Proof of Lemma 6. If $C>0$ is the best constant in (14), then on the basis of principle of duality in l_q we have

$$\begin{aligned}C&=\sup_{g\geq 0}\frac{\left\|S_r^*g\right\|_{q,\widetilde{u}}}{\left\|g\right\|_{p,v}}=\sup_{g\geq 0}\sup_{f\geq 0}\frac{\sum_{k=1}^{\infty}f_k(S_r^*g)_k\widetilde{u}_k^q}{\left\|f\right\|_{q',\widetilde{u}^{q-1}}\left\|g\right\|_{p,v}}\\&=\sup_{g\geq 0}\sup_{f\geq 0}\frac{\sum_{i=1}^{\infty}g_i(S_rf\widetilde{u}^q)_i}{\left\|g\right\|_{p,v}\left\|f\right\|_{q',\widetilde{u}^{q-1}}}=\sup_{f\geq 0}\frac{\left\|S_rf\widetilde{u}^q\right\|_{p',v^{-1}}}{\left\|f\right\|_{q',\widetilde{u}^{q-1}}}.\end{aligned} \tag{16}$$

□

Lemma 7. *Let $1<q<p<\infty$ and $0\leq r\leq n-1$. If inequality (14) holds with the least constant $C>0$, then*

$$B_r(n)\leq C. \tag{17}$$

Proof of Lemma 7. Suppose that inequality (14) holds. Then due to Lemma 6 inequality (15) also holds with the same best constant as in (14).

Let $t\geq x\geq 1$. Introduce the test sequence $f=\{f_j\}_{j=1}^{\infty}$ as follows: $f_j=0$ for $j>x$ and $f_j=(\sum_{i=1}^{j}\widetilde{u}_i^q)^{(q-1)/(p-q)}(\sum_{k=j}^{t}A_{r,1}^{p'}(k,j)v_k^{-p'})^{(p-1)(q-1)/(p-q)}$ for $1\leq j\leq x$.

If we substitute this sequence in the right side of (15), we have

$$\begin{aligned}&\left\|f\right\|_{q',\widetilde{u}^{q-1}}\\&=\left(\sum_{j=1}^{x}\left(\sum_{i=1}^{j}\widetilde{u}_i^q\right)^{q/(p-q)}\right.\\&\qquad\left.\times\left(\sum_{k=j}^{t}A_{r,1}^{p'}(k,j)\,v_k^{-p'}\right)^{q(p-1)/(p-q)}\widetilde{u}_j^q\right)^{1/q'}.\end{aligned} \tag{18}$$

Now we will work with the left side of (15) and the test sequence:

$$\begin{aligned}&\left\|S_rf\widetilde{u}^q\right\|_{p',v^{-1}}^{p'}\\&=\sum_{i=1}^{\infty}v_i^{-p'}\left(\sum_{j=1}^{i}A_{r,1}(i,j)\,f_j\widetilde{u}_j^q\right)\left(\sum_{k=1}^{i}A_{r,1}(i,k)\,f_k\widetilde{u}_k^q\right)^{p'-1}\end{aligned} \tag{19}$$

(reduce the sum in the second brackets)

$$\geq\sum_{i=1}^{\infty}v_i^{-p'}\sum_{j=1}^{i}A_{r,1}(i,j)\,f_j\widetilde{u}_j^q\left(\sum_{k=1}^{j}A_{r,1}(i,k)\,f_k\widetilde{u}_k^q\right)^{p'-1} \tag{20}$$

(change order of summation)

$$=\sum_{j=1}^{\infty}f_j\sum_{i=j}^{\infty}A_{r,1}(i,j)\,v_i^{-p'}\left(\sum_{k=1}^{j}A_{r,1}(i,k)\,f_k\widetilde{u}_k^q\right)^{p'-1}\widetilde{u}_j^q \tag{21}$$

(use nonincreasing of $A_{r,1}(i,k)$ in the second argument)

$$\geq \sum_{j=1}^{\infty} f_j \sum_{i=j}^{\infty} A_{r,1}^{p'}(i,j)\, v_i^{-p'} \left(\sum_{k=1}^{j} f_k \widetilde{u}_k^q \right)^{p'-1} \widetilde{u}_j^q \tag{22}$$

(substitute the test sequence)

$$\begin{aligned} \geq \sum_{j=1}^{x} & \left(\sum_{i=1}^{j} \widetilde{u}_i^q \right)^{(q-1)/(p-q)} \left(\sum_{k=j}^{t} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right)^{(p-1)(q-1)/(p-q)} \\ & \times \sum_{i=j}^{t} A_{r,1}^{p'}(i,j)\, v_i^{-p'} \\ & \times \left(\sum_{k=1}^{j} \left(\sum_{i=1}^{k} \widetilde{u}_i^q \right)^{(q-1)/(p-q)} \right. \\ & \times \left. \left(\sum_{i=k}^{t} A_{r,1}^{p'}(i,k)\, v_i^{-p'} \right)^{(p-1)(q-1)/(p-q)} \widetilde{u}_k^q \right)^{p'-1} \widetilde{u}_j^q \end{aligned} \tag{23}$$

(join together the similar cofactors)

$$\begin{aligned} \geq \sum_{j=1}^{x} & \left(\sum_{i=1}^{j} \widetilde{u}_i^q \right)^{(q-1)/(p-q)} \left(\sum_{k=j}^{t} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right)^{q(p-1)/(p-q)} \\ & \times \left(\sum_{k=1}^{j} \left(\sum_{i=1}^{k} \widetilde{u}_i^q \right)^{(q-1)/(p-q)} \widetilde{u}_k^q \right)^{p'-1} \widetilde{u}_j^q \end{aligned} \tag{24}$$

(use (8))

$$\begin{aligned} \gg \sum_{j=1}^{x} & \left(\sum_{i=1}^{j} \widetilde{u}_i^q \right)^{(q-1)/(p-q)} \left(\sum_{k=j}^{t} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right)^{q(p-1)/(p-q)} \\ & \times \left(\sum_{k=1}^{j} \Delta_k^{-} \left(\sum_{i=1}^{k} \widetilde{u}_i^q \right)^{(p-1)/(p-q)} \right)^{p'-1} \widetilde{u}_j^q \end{aligned} \tag{25}$$

(simplify)

$$\begin{aligned} = \sum_{j=1}^{x} & \left(\sum_{k=j}^{t} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right)^{q(p-1)/(p-q)} \\ & \times \left(\sum_{i=1}^{j} \widetilde{u}_i^q \right)^{q/(p-q)} \widetilde{u}_j^q. \end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned} \left\| S_r f \widetilde{u}^q \right\|_{p',v^{-1}}^{p'} \gg & \left(\sum_{j=1}^{x} \left(\sum_{k=j}^{t} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ & \left. \times \left(\sum_{i=1}^{j} \widetilde{u}_i^q \right)^{q/(p-q)} \widetilde{u}_j^q \right)^{1/p'}, \end{aligned} \tag{27}$$

which, together with (15) and (18), gives

$$\begin{aligned} C \gg & \left(\sum_{j=1}^{x} \left(\sum_{k=j}^{t} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ & \left. \times \left(\sum_{i=1}^{j} \widetilde{u}_i^q \right)^{q/(p-q)} \widetilde{u}_j^q \right)^{(p-q)/pq}, \end{aligned} \tag{28}$$

for all $t \geq x \geq 1$.

Now in (28) we approach $t \to \infty$ and use (8)

$$\begin{aligned} C \gg & \left(\sum_{j=1}^{x} \left(\sum_{k=j}^{\infty} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ & \left. \times \Delta_j^{-} \left(\sum_{i=1}^{j} \widetilde{u}_i^q \right)^{p/(p-q)} \right)^{(p-q)/pq} \end{aligned} \tag{29}$$

(then we use the Abelian transformation)

$$\begin{aligned} = & \left(\sum_{j=1}^{x} \left(\sum_{i=1}^{j} \widetilde{u}_i^q \right)^{p/(p-q)} \Delta_j^{+} \left(\sum_{k=j}^{\infty} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ & + \left(\sum_{i=1}^{x} \widetilde{u}_i^q \right)^{p/(p-q)} \\ & \left. \times \left(\sum_{k=x+1}^{\infty} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right)^{q(p-1)/(p-q)} \right)^{(p-q)/pq} \end{aligned} \tag{30}$$

(and use (7))

$$\begin{aligned} \gg & \left(\sum_{j=1}^{x} \left(\sum_{i=1}^{j} \widetilde{u}_i^q \right)^{p/(p-q)} \left(\sum_{k=j}^{\infty} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right)^{p(q-1)/(p-q)} \right. \\ & \left. \times \Delta_j^{+} \left(\sum_{k=j}^{\infty} A_{r,1}^{p'}(k,j)\, v_k^{-p'} \right) \right)^{(p-q)/pq}. \end{aligned} \tag{31}$$

Substituting $\widetilde{u}_i^q$ and approaching $x \to \infty$, we obtain (17). □

Lemma 8. *Let $1 < q < \infty$ and $0 \leq r \leq n-1$. Then*

$$\left\| S_r^* f \right\|_{q,\widetilde{u}} \ll \left\| S_{n-1}^* f \right\|_{q,u}, \quad f \geq 0. \tag{32}$$

Proof of Lemma 8. The expression $A_{n-1,1}(i,j)$ can be transformed as follows:

$$A_{n-1,1}(i,j) = \sum_{k_{n-1}=j}^{i} \omega_{n-1,k_{n-1}} \cdots \sum_{k_{r+1}=k_{r+2}}^{i} \omega_{r+1,k_{r+1}} \times \sum_{k_r=k_{r+1}}^{i} \omega_{r,k_r} \cdots \sum_{k_1=k_2}^{i} \omega_{1,k_1} = \sum_{k_{n-1}=j}^{i} \omega_{n-1,k_{n-1}} \cdots \sum_{k_{r+2}=k_{r+3}}^{i} \omega_{r+2,k_{r+2}} \times \sum_{k_{r+1}=k_{r+2}}^{i} \omega_{r+1,k_{r+1}} A_{r,1}(i,k_{r+1}) \quad (33)$$

(change of order of summation)

$$= \sum_{k_{r+1}=j}^{i} A_{r,1}(i,k_{r+1}) \omega_{r+1,k_{r+1}} \times \sum_{k_{n-1}=j}^{k_{r+1}} \omega_{n-1,k_{n-1}} \cdots \sum_{k_{r+2}=k_{r+3}}^{k_{r+1}} \omega_{r+2,k_{r+2}} = \sum_{t=j}^{i} A_{r,1}(i,t)\, \omega_{r+1,t} A_{n-1,r+2}(t,j). \quad (34)$$

Since $\Delta_t^- A_{n-1,r+1}(t,j) = \omega_{r+1,t} A_{n-1,r+2}(t,j)$, then

$$A_{n-1}(i,j) = \sum_{t=j}^{i} A_{r,1}(i,t)\, \Delta_t^- A_{n-1,r+1}(t,j). \quad (35)$$

Let $f \geq 0$; then

$$\|S_{n-1}^* f\|_{q,u}^q = \sum_{j=1}^{\infty} u_j^q \left(\sum_{i=j}^{\infty} A_{n-1,1}(i,j) f_i \right)^{q-1} \times \sum_{k=j}^{\infty} A_{n-1,1}(k,j) f_k \geq \sum_{j=1}^{\infty} u_j^q \sum_{k=j}^{\infty} A_{n-1,1}(k,j) f_k \times \left(\sum_{i=k}^{\infty} A_{n-1,1}(i,j) f_i \right)^{q-1} \quad (36)$$

(use (35))

$$= \sum_{j=1}^{\infty} u_j^q \sum_{k=j}^{\infty} f_k \sum_{t=j}^{k} A_{r,1}(k,t)\, \Delta_t^- A_{n-1,r+1}(t,j) \times \left(\sum_{i=k}^{\infty} A_{n-1,1}(i,j) f_i \right)^{q-1} \quad (37)$$

(change order of summation)

$$= \sum_{j=1}^{\infty} u_j^q \sum_{t=j}^{\infty} \Delta_t^- A_{n-1,r+1}(t,j) \times \sum_{k=t}^{\infty} A_{r,1}(k,t) f_k \left(\sum_{i=k}^{\infty} A_{n-1,1}(i,j) f_i \right)^{q-1} \quad (38)$$

(use $A_{n-1,1}(i,j) \geq A_{r,1}(i,t) A_{n-1,r+1}(t,j)$, which follows from (6))

$$\geq \sum_{j=1}^{\infty} u_j^q \sum_{t=j}^{\infty} (\Delta_t^- A_{n-1,r+1}(t,j))\, A_{n-1,r+1}^{q-1}(t,j) \times \sum_{k=t}^{\infty} A_{r,1}(k,t) f_k \left(\sum_{i=k}^{\infty} A_{r,1}(i,t) f_i \right)^{q-1} \quad (39)$$

(use (7) and (8))

$$\gg \sum_{j=1}^{\infty} u_j^q \sum_{t=j}^{\infty} \Delta_t^- A_{n-1,r+1}^q(t,j) \sum_{k=t}^{\infty} \Delta_k^+ \left(\sum_{i=k}^{\infty} A_{r,1}(i,t) f_i \right)^q \quad (40)$$

(use the relation $\Delta_k^+ (\sum_{i=k}^{\infty} A_{r,1}(i,t) f_i) = A_{r,1}(k,t) f_k$)

$$= \sum_{j=1}^{\infty} u_j^q \sum_{t=j}^{\infty} \Delta_t^- A_{n-1,r+1}^q(t,j) \left(\sum_{i=t}^{\infty} A_{r,1}(i,t) f_i \right)^q \quad (41)$$

(rearrange the cofactors)

$$= \sum_{t=1}^{\infty} \left(\sum_{i=t}^{\infty} A_{r,1}(i,t) f_i \right)^q \sum_{j=1}^{t} \Delta_t^- A_{n-1,r+1}^q(t,j)\, u_j^q = \quad (42)$$

(take into account that $A_{n-1,r+1}(t-1,t) = 0$)

$$= \sum_{t=1}^{\infty} \left(\sum_{i=t}^{\infty} A_{r,1}(i,t) f_i \right)^q \Delta_t^- \left(\sum_{j=1}^{t} A_{n-1,r+1}^q(t,j)\, u_j^q \right) = \|S_r^* f\|_{q,\tilde{u}}; \quad (43)$$

that is, we have (32). □

4. Proofs of Theorems 4 and 5

First we prove Theorem 5.

Proof of Theorem 5

Necessity. Suppose that operator (5) is bounded from $l_{p,v}$ to $l_{q,u}$ that equivalently means the validity of the following inequality:

$$\|S_{n-1}^* f\|_{q,u} \leq C \|f\|_{p,v}, \quad f \geq 0. \quad (44)$$

Then due to (32) inequality (14) holds for any $r = 0, 1, \ldots, n-1$. Therefore, by Lemma 7 we have $B_r(n) \ll C$, $0 \leq r \leq n-1$, that means $B(n) \ll C$, where C is the best constant in (44); that is, $C = \| S_{n-1}^* \|$.

Sufficiency. will be proved by the induction method.

For $n = 1$ the operator $S^*_{n-1} = S^*_0$ is the Hardy operator. Thus by Lemma 3 if $B_0(1) < \infty$, then the operator S^*_0 is bounded from $l_{p,v}$ to $l_{q,u}$ with the estimate $\| S^*_0 \| \ll B_0(1)$.

Next we assume that for $n = 1, 2, \ldots, l$, $1 \le l < n-1$, if $B(n) < \infty$, then the operator S^*_{n-1}, $n = 1, 2, \ldots, l$, is bounded from $l_{p,v}$ to $l_{q,u}$ with the estimate $\| S^*_{n-1} \| \ll B(n)$.

Now we need to prove that for $n = l+1$ if $B(l+1) = \max_{0\le r\le l} B_r(l+1) < \infty$, then the operator S^*_l is bounded from $l_{p,v}$ to $l_{q,u}$ with the estimate $\| S^*_l \| \ll B(l+1)$.

Let $f \ge 0$ and $T_j = \{k \in \mathbb{Z} : 2^{-k} \le (S^*_l f)_j\}$, $j \in \mathbb{N}$. Suppose that $k_j = \inf T_j$ for $T_j \ne \emptyset$ and $k_j = \infty$ for $T_j = \emptyset$. When $k_j < \infty$, then $2^{-k_j} \le (S^*_l f)_j < 2^{-k_j+1}$. Further, for exception of the trivial case we suppose that $(S^*_l f)_1 > 0$.

Let $m_1 = 1$ $M_1 = \{j \in \mathbb{N} : k_j = k_{m_1}\}$. Assume that $\sup M_1 + 1 = m_2$. If $\sup M_1 < \infty$, then $m_2 < \infty$ and $\sup M_1 = \max M_1 = m_2 - 1$. It is obvious that $m_1 < m_2$. Suppose that we have found $m_1 < m_2 < \cdots < m_s < \infty$, $s > 1$. Then we determine m_{s+1} as $m_{s+1} = \sup M_s + 1$, where $M_s = \{j \in \mathbb{N} : k_j = k_{m_s}\}$. By the definition $2^{-k_{m_s}} \le (S^*_l f)_j < 2^{-k_{m_s}+1}$, $m_s \le j \le m_{s+1} - 1$, and $k_{m_s} < k_{m_{s+1}}$, $s \ge 1$.

Further, for convenience let $k_{m_s} = n_s$ and $m_{s+1} - 1 = m'_{s+1}$; then

$$2^{-n_s} \le (S^*_l f)_j < 2^{-n_s+1}, \quad m_s \le j \le m'_{s+1}, \tag{45}$$

and $n_s < n_{s+1}$, $s \ge 1$.

Let $\mathbb{N}_0 = \{s \in \mathbb{N} : m_s < \infty\}$. Then from (45) it follows

$$\mathbb{N} = \bigcup_{s\in\mathbb{N}_0} [m_s, m_{s+1}). \tag{46}$$

There are two possible cases: (1) $\mathbb{N}_0 = \mathbb{N}$ and (2) $\mathbb{N}_0 = \{1, 2, \ldots, s_0\}$, $s_0 > 1$.

Case (1) $\mathbb{N}_0 = \mathbb{N}$. Since $n_s < n_{s+1}$, $s \ge 1$, then $-n_s - 1 \le -n_{s+2} + 1$. Therefore, using (45) and (6), we have

$$\begin{aligned}
2^{-n_s-1} &= 2^{-n_s} - 2^{-n_s-1} \le 2^{-n_s} - 2^{-n_{s+2}+1} \\
&\le (S^*_l f)_{m'_{s+1}} - (S^*_l f)_{m_{s+2}} = \sum_{i=m'_{s+1}}^{m'_{s+2}} A_{l,1}(i, m'_{s+1}) f_i \\
&\quad + \sum_{i=m_{s+2}}^{\infty} \left[A_{l,1}(i, m'_{s+1}) - A_{l,1}(i, m_{s+2})\right] f_i \\
&\le \sum_{i=m'_{s+1}}^{m'_{s+2}} A_{l,1}(i, m'_{s+1}) f_i + \sum_{i=0}^{l-1} A_{l,r+1}(m_{s+2}, m'_{s+1}) \\
&\quad \times \sum_{i=m_{s+2}}^{\infty} A_{r,1}(i, m_{s+2}) f_i.
\end{aligned} \tag{47}$$

Using (45), (46), and (47), we get

$$\begin{aligned}
\|S^*_l f\|^q_{q,u} &= \sum_{s\ge1} \sum_{i=m_s}^{m'_{s+1}} u^q_i (S^*_l f)^q_i < \sum_s \left(2^{-n_s+1}\right)^q \sum_{i=m_s}^{m'_{s+1}} u^q_i \\
&= 2^{2q} \sum_s \left(2^{-n_s-1}\right)^q \sum_{i=m_s}^{m'_{s+1}} u_i \\
&\ll \sum_s \left(\sum_{i=m'_{s+1}}^{m'_{s+2}} A_{l,1}(i, m'_{s+1}) f_i \right)^q \sum_{i=m_s}^{m'_{s+1}} u^q_i \\
&\quad + \sum_{r=0}^{l-1} \sum_s A^q_{l,r+1}(m_{s+2}, m'_{s+1}) \\
&\quad \times \left(\sum_{i=m_{s+2}}^{\infty} A_{r,1}(i, m_{s+2}) f_i \right)^q \sum_{i=m_s}^{m'_{s+1}} u^q_i = I_l + \sum_{i=0}^{l-1} I_r.
\end{aligned} \tag{48}$$

First we estimate I_l. Using Hölder's inequality twice, we have

$$\begin{aligned}
I_l &\le \sum_s \left(\sum_{i=m'_{s+1}}^{m'_{s+2}} A^{p'}_{l,1}(i, m'_{s+1}) v_i^{-p'} \right)^{q/p'} \\
&\quad \times \sum_{i=m_s}^{m'_{s+1}} u^q_i \left(\sum_{i=m'_{s+1}}^{m'_{s+2}} (v_i f_i)^p \right)^{q/p} \\
&\le \left(\sum_s \left(\sum_{i=m'_{s+1}}^{m'_{s+2}} A^{p'}_{l,1}(i, m'_{s+1}) v_i^{-p'} \right)^{q(p-1)/(p-q)} \right. \\
&\quad \left. \times \left(\sum_{i=m_s}^{m'_{s+1}} u^q_i \right)^{p/(p-q)} \right)^{(p-q)/p} \\
&\quad \times \left(\sum_s \sum_{i=m'_{s+1}}^{m'_{s+2}} (v_i f_i)^p \right)^{q/p} \ll \widetilde{B}_l^{(p-q)/p} \|f\|^q_{p,v},
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
\widetilde{B}_l &= \sum_s \left(\sum_{i=m'_{s+1}}^{m'_{s+2}} A^{p'}_{l,1}(i, m'_{s+1}) v_i^{-p'} \right)^{q(p-1)/(p-q)} \\
&\quad \times \left(\sum_{i=m_s}^{m'_{s+1}} u^q_i \right)^{p/(p-q)}.
\end{aligned} \tag{50}$$

Next we need the following obvious inequality:

$$\begin{aligned}\Delta_i^+\left(\sum_{j=i}^{\infty}A_{l,1}^{p'}(j,i)\,v_j^{-p'}\right) &= A_{l,1}^{p'}(i,i)+\sum_{j=i+1}^{\infty}\Delta_i^+A_{l,1}^{p'}(j,i)\,v_j^{-p'}\\ &\geq A_{l,1}^{p'}(i,i)+\sum_{j=i+1}^{m'_{s+2}}\Delta_i^+A_{l,1}^{p'}(j,i)\,v_j^{-p'}\\ &= \Delta_i^+\left(\sum_{j=i}^{m'_{s+2}}A_{l,1}^{p'}(j,i)\,v_j^{-p'}\right),\quad m'_{s+1}\leq i\leq m'_{s+2}.\end{aligned}\tag{51}$$

Now, taking into account the above inequality (51) and using (10) and (7), we estimate $\widetilde{B}_l$:

$$\begin{aligned}\widetilde{B}_l &\ll \sum_s\sum_{i=m'_{s+1}}^{m'_{s+2}}\left(\sum_{j=i}^{m'_{s+2}}A_{l,1}^{p'}(j,i)\,v_j^{-p'}\right)^{p(q-1)/(p-q)}\\ &\quad\times\left(\sum_{k=m_s}^{m'_{s+1}}u_k^q\right)^{p/(p-q)}\Delta_i^+\left(\sum_{j=i}^{m'_{s+2}}A_{l,1}^{p'}(j,i)\,v_j^{-p'}\right)\\ &\leq\sum_s\sum_{i=m'_{s+1}}^{m'_{s+2}}\left(\sum_{j=i}^{\infty}A_{l,1}^{p'}(j,i)\,v_j^{-p'}\right)^{p(q-1)/(p-q)}\\ &\quad\times\left(\sum_{k=1}^{i}u_k^q\right)^{p/(p-q)}\Delta_i^+\left(\sum_{j=i}^{\infty}A_{l,1}^{p'}(j,i)\,v_j^{-p'}\right)\\ &\ll\sum_{i=1}^{\infty}\left(\sum_{j=i}^{\infty}A_{l,1}^{p'}(j,i)\,v_j^{-p'}\right)^{p(q-1)/(p-q)}\\ &\quad\times\left(\sum_{k=1}^{i}u_k^q\right)^{p/(p-q)}\Delta_i^+\left(\sum_{j=i}^{\infty}A_{l,1}^{p'}(j,i)\,v_j^{-p'}\right)\\ &=\left(B_l(l+1)\right)^{pq/(p-q)}.\end{aligned}\tag{52}$$

From (49) and (52) we have

$$I_l\ll B_l^q(l+1)\,\|f\|_{p,v}^q\leq B^q(l+1)\,\|f\|_{p,v}^q.\tag{53}$$

Now we estimate I_r, $r=0,1,\ldots,l-1$. Let $\mathbb{N}_1=\{k=m_{s+2}:s\in\mathbb{N}\}$.

Assume that $\Delta_k^q=A_{l,r+1}^q(m_{s+2},m'_{s+1})\sum_{i=m_s}^{m'_{s+1}}u_i^q$ if $k=m_{s+2}\in\mathbb{N}_1$ and $\Delta_k^q=0$ if $k\notin\mathbb{N}_1$. Then

$$I_r=\sum_{k=1}^{\infty}\Delta_k^q\left(\sum_{i=k}^{\infty}A_{r,1}(i,k)\,f_i\right)^q=\|S_r^*f\|_{q,\Delta}^q,\quad r=0,1,\ldots,l-1.\tag{54}$$

The operator S_r^* is the operator S_{n-1}^* for $n=r+1$ and $1\leq r+1\leq l$. Therefore, by our assumption we have $\|S_r^*\|\ll\widetilde{B}(r+1)=\max_{0\leq t\leq r}\widetilde{B}_t(r+1)$. Hence,

$$I_r\ll\max_{0\leq t\leq r}\widetilde{B}_t(r+1),\tag{55}$$

where

$$\begin{aligned}\left(\widetilde{B}_t(r+1)\right)^{pq/(p-q)} &= \sum_{i=1}^{\infty}\left(\sum_{j=i}^{\infty}A_{t,1}^{p'}(j,i)\,v_j^{-p'}\right)^{p(q-1)/(p-q)}\\ &\quad\times\left(\sum_{k=1}^{i}A_{r,t+1}^q(i,k)\,\Delta_k^q\right)^{p/(p-q)}\\ &\quad\times\Delta_i^+\left(\sum_{j=i}^{\infty}A_{t,1}^{p'}(j,i)\,v_j^{-p'}\right).\end{aligned}\tag{56}$$

We estimate the expression $\sum_{k=1}^{i}A_{r,t+1}^q(i,k)\Delta_k^q$:

$$\begin{aligned}&\sum_{k=1}^{i}A_{r,t+1}^q(i,k)\,\Delta_k^q\\ &\quad=\sum_{m_{s+2}\leq i}A_{r,t+1}^q(i,m_{s+2})\,A_{l,r+1}^q(m_{s+2},m'_{s+1})\sum_{j=m_s}^{m'_{s+1}}u_j^q\end{aligned}\tag{57}$$

(use the left side of (6) and nonincreasing of $A_{l,t+1}^q(i,j)$ in the second argument)

$$\begin{aligned}&\leq\sum_{m_{s+2}\leq i}A_{l,t+1}^q(i,m'_{s+1})\sum_{j=m_s}^{m'_{s+1}}u_j^q\leq\sum_{m_{s+2}\leq i}\sum_{j=m_s}^{m'_{s+1}}A_{l,t+1}^q(i,j)\,u_j^q\\ &\leq\sum_{j=1}^{i}A_{l,t+1}^q(i,j)\,u_j^q.\end{aligned}\tag{58}$$

Substituting the obtained estimate in (56), we have $\widetilde{B}_t(r+1)\leq B_t(l+1)$. Therefore, from (55) we get

$$I_r\ll\max_{0\leq t\leq r}B_t^q(l+1)\,\|f\|_{p,v}^q\leq B^q(l+1)\,\|f\|_{p,v}^q,\quad r=0,1,\ldots,l-1.\tag{59}$$

Then from (48), (53), and (59) we obtain

$$\|S_l^*f\|_{q,u}\ll B(l+1)\,\|f\|_{p,v},\tag{60}$$

$$\|S_l^*\|\ll B(l+1).\tag{61}$$

Now we turn to case (2) $\mathbb{N}_0=\{1,2,\ldots,s_0\}$, $1\leq s_0<\infty$. In this case we have $m_{s_0}<\infty$ and $m_{s_0+1}=\infty$. Here there are two possible cases: $n_{s_0}<\infty$ and $n_{s_0}=\infty$.

Below we suppose that $\sum_{s=k}^{t}=\sum_{s=1}^{t}$ if $k<0$.

Let $n_{s_0} < \infty$. Then

$$\begin{aligned}\|S_l^* f\|_{q,u}^q &= \sum_{s=1}^{s_0} \sum_{j=m_s}^{m_{s+1}} (S_l^* f)_j^q u_j \\ &= \sum_{s=1}^{s_0-2} \sum_{j=m_s}^{m'_{s+1}} (S_l^* f)_j^q u_j^q + \sum_{j=m_{s_0-1}}^{m'_{s_0}} (S_l^* f)_j^q u_j^q \\ &\quad + \sum_{j=m_{s_0}}^{\infty} (S_l^* f)_j^q u_j^q = J_1 + J_2 + J_3.\end{aligned} \tag{62}$$

If $J_1 \neq 0$, then $s_0 > 2$, and as in the case $\mathbb{N}_0 = \mathbb{N}$ using (47), we estimate J_1 and as a result we get

$$J_1 \ll B^q (l+1) \|f\|_{p,v}^q. \tag{63}$$

Using (45) and Hölder's inequality, we estimate the value J_2:

$$\begin{aligned}J_2 &< 2^q (2^{-n_{s_0-1}})^q \sum_{j=m_{s_0-1}}^{m'_{s_0}} u_j^q \\ &\ll \left(\sum_{i=m'_{s_0}}^{\infty} A_{l,1}(i, m'_{s_0}) f_i \right)^q \sum_{j=m_{s_0-1}}^{m'_{s_0}} u_j^q \\ &\leq \left(\sum_{i=m'_{s_0}}^{\infty} A_{l,1}^{p'}(i, m'_{s_0}) v_i^{-p'} \right)^{q/p'} \\ &\quad \times \sum_{j=m_{s_0-1}}^{m'_{s_0}} u_j^q \left(\sum_{i=m'_{s_0}}^{\infty} |v_i f_i|^p \right)^{q/p} \\ &\leq \left[\left(\sum_{i=m'_{s_0}}^{\infty} A_{l,1}^{p'}(i, m'_{s_0}) v_i^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ &\quad \left. \times \left(\sum_{j=m_{s_0-1}}^{m'_{s_0}} u_j^q \right)^{p/(p-q)} \right]^{(p-q)/p} \|f\|_{p,v}^q.\end{aligned} \tag{64}$$

Using (7) and (10), we estimate the expression in the square brackets:

$$\begin{aligned}&\left(\sum_{i=m'_{s_0}}^{\infty} A_{l,1}^{p'}(i, m'_{s_0}) v_i^{-p'} \right)^{q(p-1)/(p-q)} \left(\sum_{j=m_{s_0-1}}^{m'_{s_0}} u_j^q \right)^{p/(p-q)} \\ &\ll \sum_{i=m'_{s_0}}^{\infty} \left(\sum_{j=i}^{\infty} A_{l,1}(j,i) v_j^{-p'} \right)^{p(q-1)/(p-q)} \\ &\quad \times \Delta_i^+ \left(\sum_{j=i}^{\infty} A_{l,1}(j,i) v_j^{-p'} \right) \left(\sum_{j=m_{s_0-1}}^{m'_{s_0}} u_j^q \right)^{p/(p-q)} \\ &\leq \sum_{i=m'_{s_0}}^{\infty} \left(\sum_{j=i}^{\infty} A_{l,1}(j,i) v_j^{-p'} \right)^{p(q-1)/(p-q)} \\ &\quad \times \left(\sum_{j=1}^{i} u_j^q \right)^{p/(p-q)} \Delta_i^+ \left(\sum_{j=i}^{\infty} A_{l,1}(j,i) v_j^{-p'} \right) \\ &\leq (B_l(l+1))^{pq/(p-q)}.\end{aligned} \tag{65}$$

Therefore,

$$J_2 \ll (B_l(l+1))^q \|f\|_{p,v}^q \leq B^q(l+1) \|f\|_{p,v}^q. \tag{66}$$

Now we estimate J_3. Since $2^{-n_{s_0}} \leq (S_l^* f)_j < 2^{-n_{s_0}+1}$ for all $j \geq m_{s_0}$, then $(S_l^* f)_j \leq 2(S_l^* f)_t$ for all $j, t \geq m_{s_0}$. Hence,

$$\begin{aligned}J_3 &\leq \sup_{t \geq m_{s_0}} \sum_{j=m_{s_0}}^{t} (S_l^* f)_j^q u_j^q \\ &\leq 2^q \sup_{t \geq m_{s_0}} \left(\sum_{i=t}^{\infty} A_{n-1,1}(i,t) f_i \right)^q \sum_{j=m_{s_0}}^{t} u_j^q.\end{aligned} \tag{67}$$

Further, as for the estimate of J_2 by Hölder's inequality, we have

$$\begin{aligned}J_3 &\ll \sup_{t \geq m_{s_0}} \left[\left(\sum_{i=t}^{\infty} A_{l,1}^q(i,t) v_i^{-p'} \right)^{q(p-1)/(p-q)} \right. \\ &\quad \left. \times \left(\sum_{j=m_{s_0}}^{t} u_j^q \right)^{p/(p-q)} \right]^{(p-q)/p} \|f\|_{p,v}^q \\ &\ll B_l^q(l+1) \|f\|_{p,v}^q \leq B^q(l+1) \|f\|_{p,v}^q.\end{aligned} \tag{68}$$

From (62), (63), (66), and (68) we have (60) and (61). If $n_{s_0} = \infty$ that means $k_{m_s} = \infty$, then $T_j = \emptyset$ for $j \geq m_{s_0}$; that is, $(S_l^* f)_j = 0$ for $j \geq m_{s_0}$. By the assumption $(S_l^* f)_1 > 0$, therefore, $s_0 > 1$. Then $m_2 < \infty$ and $s_0 \geq 2$. Hence,

$$\|S_l^* f\|_{q,u}^q = \sum_{s=1}^{s_0-2} \sum_{j=m_s}^{m'_{s+1}} (S_l^* f)_j^q u_j^q + \sum_{j=m_{s_0-1}}^{m'_{s_0}} (S_n^* f)_j^q u_j^q = J_1 + J_2. \tag{69}$$

This, together with estimates (63) and (66), gives (60) and (61). Thus, if $B(l+1) < \infty$, then the operator S_l^* is bounded from $l_{p,v}$ to $l_{q,u}$ and estimate (61) holds. Consequently, for any