Springer
Monographs in
Mathematics

P. Tauvel
R. W. T. Yu

Lie Algebras and Algebraic Groups

李代数和代数群

Springer

世界图书出版公司
www.wpcbj.com.cn

Patrice Tauvel
Rupert W. T. Yu

Lie Algebras and Algebraic Groups

With 44 Figures

Springer

Patrice Tauvel
Rupert W. T. Yu
Département de Mathématiques
Université de Poitiers
Boulevard Marie et Pierre Curie,
Téléport 2 – BP 30179
86962 Futuroscope Chasseneuil cedex, France
e-mail: tauvel@math.univ-poitiers.fr
yuyu@math.univ-poitiers.fr

Library of Congress Control Number: 2005922400

Mathematics Subject Classification (2000): 17-01, 17-02, 17Bxx, 20Gxx

ISSN 1439-7382
ISBN-10 3-540-24170-1 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-24170-6 Springer Berlin Heidelberg New York

Springer Monographs in Mathematics

Preface

The theory of groups and Lie algebras is interesting for many reasons. In the mathematical viewpoint, it employs at the same time algebra, analysis and geometry. On the other hand, it intervenes in other areas of science, in particular in different branches of physics and chemistry. It is an active domain of current research.

One of the difficulties that graduate students or mathematicians interested in the theory come across, is the fact that the theory has very much advanced, and consequently, they need to read a vast amount of books and articles before they could tackle interesting problems.

One of the goals we wish to achieve with this book is to assemble in a single volume the basis of the algebraic aspects of the theory of groups and Lie algebras. More precisely, we have presented the foundation of the study of finite-dimensional Lie algebras over an algebraically closed field of characteristic zero.

Here, the geometrical aspect is fundamental, and consequently, we need to use the notion of algebraic groups. One of the main differences between this book and many other books on the subject is that we give complete proofs for the relationships between algebraic groups and Lie algebras, instead of admitting them.

We have also given the proofs of certain results on commutative algebra and algebraic geometry that we needed so as to make this book as self-contained as possible. We believe that in this way, the book can be useful for both graduate students and mathematicians working in this area.

Let us give a brief description of the material treated in this book.

As we have stated earlier, our goal is to study Lie algebras over an algebraically closed field of characteristic zero. This allows us to avoid, in considering questions concerning algebraic geometry, the notion of separability, which simplifies considerably our presentation. In fact, under certain conditions of separability, the correspondence between Lie algebras and algebraic groups described in chapter 24 has a very nice generalization when the algebraically closed base field has prime characteristic.

Chapters 1 to 9 treat basic results on topology, commutative algebra and sheaves of functions that are required in the rest of the book.

In chapter 10, we recall some standard results on Jordan decompositions and the theory of abstract groups and group actions. Here, the base field is assumed to be algebraically closed in order to obtain a Jordan decomposition.

Chapters 11 to 17 give an introduction to the theory of algebraic geometry which we shall encounter continually in the chapters which follow. We have selected only the notions that we require in this book. The reader should by no means consider these chapters as a thorough introduction to the theory of algebraic geometry.

Chapters 18 and 36 are dedicated to root systems which are fundamental to the study of semisimple Lie algebras.

We introduce Lie algebras in chapter 19. In this chapter, we prove important results on the structure of Lie algebras such as Engel's theorem, Lie's theorem and Cartan's criterion on solvability.

In chapter 20, we define the notions of semisimple and reductive Lie algebras. In addition to characterizing these Lie algebras, we discover in this chapter how the structure of semisimple Lie algebras can be related to root systems.

The general theory of algebraic groups is studied in chapters 21 to 28. The relations between Lie algebras and algebraic groups, which are fundamental to us, are established in chapters 23 and 24. Chapter 29 presents applications of these relations to tackle the systematic study of Lie algebras. The reader will observe that the geometrical aspects have an important part in this study. In particular, the orbits of points under the action of an algebraic group plays a central role.

Chapter 30 gives a short introduction of the theory of representations of semisimple Lie algebras which we need in order to prove Chevalley's theorem on invariants in chapter 31.

We define in chapter 32 S-triples which are essential to the study of semisimple Lie algebras. Another fundamental notion, treated in chapters 33 to 35, is the notion of nilpotent orbits in semisimple Lie algebras.

We introduce symmetric Lie algebras in chapter 37, and semisimple symmetric Lie algebras in chapter 38. In these chapters, we give generalizations of certain results of chapters 32 to 35.

In addition to presenting the essential classical results of the theory, some of the results we have included in the final chapters are recent, and some are yet to be published.

At the end of each chapter, the reader may find a list of relevant references, and in some cases, remarks concerning the contents of the chapter.

There are many approaches to reading this book. We need not read this book linearly. A reader familiar with the theory of commutative algebra may skip chapters 2 to 8, and consider these chapters for references only. Let us also point out that chapters 18, 19 and 20 constitute a short introduction to the theory of finite-dimensional Lie algebras and the structure of semisimple Lie algebras.

We wish to thank our colleagues A. Bouaziz and H. Sabourin of the university of Poitiers with whom we had many useful discussion during the preparation of this book.

Poitiers,
January 2005

Patrice Tauvel
Rupert W.T. Yu

Contents

1

Results on topological spaces

In this chapter, we treat some basic notions of topology such as irreducible and constructible sets, dimension of a topological space, Noetherian space, which are fundamental in algebraic geometry.

1.1 Irreducible sets and spaces

1.1.1 Definition. *A topological space X is said to be* irreducible *if any finite intersection of non-empty open subsets is non-empty.*

1.1.2 It follows from the definition that an irreducible topological space is not empty.

1.1.3 Proposition. *Let X be a non-empty topological space. Then the following conditions are equivalent:*

(i) *X is irreducible.*
(ii) *X is not the finite union of distinct proper closed subsets.*
(iii) *X is not the union of two proper closed subsets.*
(iv) *Any non-empty open subset of X is dense in X.*
(v) *Any open subset of X is connected.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear and (iii) $\Rightarrow$ (iv) follows from the fact that a subset in X is dense if and only if it meets all non-empty open subsets. Now if U is a non-connected non-empty open subset, then $U = U_1 \cup U_2$ where U_1, U_2 are non-empty open subsets and $U_1 \cap U_2 = \emptyset$. Thus (iv) $\Rightarrow$ (v). The same argument gives (v) $\Rightarrow$ (i). □

Remark. If X is irreducible, then it is connected. The converse is not true.

1.1.4 In the rest of this chapter, X is a topological space.

A subset of X is called *irreducible* if it is non-empty and irreducible as a topological space. From the above definitions, the following result is clear.

Proposition. *Let A be a non-empty subset of X. Then the following conditions are equivalent:*

(i) *A is irreducible.*

(ii) *Let $F_1, \ldots, F_n$ be closed subsets of X such that A is contained in the union of the F_i's, then there exists $j \in \{1, \ldots, n\}$ such that $A \subset F_j$.*

(iii) *Let U, V be open subsets of X such that $U \cap A$ and $V \cap A$ are non-empty, then $U \cap V \cap A \neq \emptyset$.*

1.1.5 Proposition. *Let A, B be subsets of X.*

(i) *A is irreducible if and only if its closure $\overline{A}$ is irreducible.*

(ii) *If A is irreducible and $A \subset B \subset \overline{A}$, then B is irreducible.*

Proof. For any open subset U, we have $U \cap A \neq \emptyset$ if and only if $U \cap \overline{A} \neq \emptyset$. So (i) and (ii) follow. □

1.1.6 Proposition. (i) *If X is irreducible, then any non-empty open subset of X is also irreducible.*

(ii) *Let $(U_i)_{i \in I}$ be a covering of X by open subsets such that $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$. If all the U_i's are irreducible, then X is irreducible.*

Proof. (i) Let U, V be non-empty open subsets of X such that $V \subset U$. If X is irreducible, then V is dense in U. Thus U is irreducible.

(ii) Let V be a non-empty open subset of X. There exists $k \in I$ such that $V \cap U_k \neq \emptyset$. Since $U_i \cap U_k \neq \emptyset$ for all $i \in I$ and $V \cap U_k$ is dense in U_k, $V \cap U_i \cap U_k \neq \emptyset$. Hence $V \cap U_i \neq \emptyset$ for all i. It follows that $V \cap U_i$ is dense in U_i for all $i \in I$, so V is dense in X. □

1.1.7 Proposition. *Let Y be a topological space and $f : X \to Y$ a continuous map.*

(i) *If $A \subset X$ is irreducible, then $f(A)$ is irreducible in Y.*

(ii) *Suppose that Y is irreducible, f is an open map and that $f^{-1}(y)$ is irreducible for all $y \in Y$. Then X is irreducible.*

Proof. (i) Let U, V be open subsets of Y such that $U \cap f(A)$ and $V \cap f(A)$ are non-empty. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets whose intersection with A is non-empty. It follows that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ meets A. Therefore $U \cap V$ meets $f(A)$ and assertion (i) follows.

(ii) Let U, V be non-empty open subsets of X. Since f is open and Y is irreducible, $f(U)$ meets $f(V)$ at some point y. Further, $f^{-1}(y)$ is irreducible, therefore the open subsets $U \cap f^{-1}(y)$ and $V \cap f^{-1}(y)$ of $f^{-1}(y)$ have non-empty intersection. Hence $U \cap V \neq \emptyset$. □

1.1.8 Remark. A map $f : X \to Y$ is called *dominant* if $f(X)$ is dense in Y. It follows from 1.1.5 and 1.1.7 that if X is irreducible and f is continuous and dominant, then Y is irreducible.