GRAPHICAL ENUMERATION

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Enumerate, count, number, call over, run over, take an account of, call the roll, muster, poll, sum up, cast up, tell off, cipher, reckon, reckon up, estimate, compute, calculate.

Roget, "*Thesaurus*"

PREFACE

The first question asked by many students in a course in graph theory is "How many graphs are there?" This is also the first problem we attempted. As circumstances had it, we learned by a most circuitous procedure that George Pólya had already counted graphs with a given number of points and lines. Starting from his formulas, it was a relatively routine matter to enumerate rooted graphs, connected graphs, and directed graphs. Subsequently, we counted various other types of graphs and when we had temporarily exhausted all the easy counting problems, we published a paper presenting 27 unsolved enumeration problems. By now, almost half of these problems have been resolved, and successive revisions of the original list of 27 unsolved enumeration problems were prepared. Our closing chapter brings this topic up to date.

Although Euler counted certain types of triangulated polygons in the plane, the major activity in graphical enumeration was launched in the preceding century. Cayley counted three types of trees: labeled trees, rooted trees,and ordinary trees. Even earlier, the world's first electrical engineer, Kirchhoff, implicitly had found the number of spanning trees in a given connected graph, and thus in particular, the number of labeled trees. In one of the earliest instances of support of combinatorial research by the military (aside from Archimedes), Major P. A. MacMahon wrote a comprehensive treatise that touched on graphical enumeration, but only peripherally. There is another pre-Pólya innovator in the art of combinatorial enumeration. This largely unsung hero, J. Howard Redfield, wrote exactly one paper on the subject; in it he anticipated many of the counting methods and results found subsequently. His paper went almost completely unrecognized. Long after Pólya's great work served as the impetus for most of the contemporary research on the counting of graphs, proper acknowledgment to Redfield was accorded.

Although we are restricting ourselves to the enumeration of various kinds of graphs, there are many types of configurations that can be so handled. The following structures, none of which is blatantly graphical at first blush, have all been enumerated by clever transformations into graphs or subgraphs: automata, finite topologies, boolean functions, necklaces, and chemical isomers.

It is not only true that a full book can be written on each of our ten chapters, but a fortiori, an entire book has been written on one of the sections of our first chapter: a formal but comprehensive monograph entitled "Counting Labeled Trees" by John Moon. Clearly the material to be included in each chapter must necessarily be a matter of personal taste.

The plan of the book is as follows. We begin with labeled graphs in Chapter 1, both in order to get them out of the way and because they are much easier to count. We then develop the basic enumeration theorem of Pólya in Chapter 2. With this available, we count in Chapter 3 an enormous variety of trees and then in Chapters 4 and 5 various kinds of graphs and digraphs. Chapter 6 presents the powerful Power Group Enumeration Theorem and shows how to apply it. Chapter 7, Superposition, counts those configurations that can be constructed by "plopping things on top of other things." Nonseparable graphs, also known as blocks, are then counted in Chapter 8 using the ingenious methods conceived by the hero of unsolved enumeration problems, R. W. Robinson. Some mathematicians feel that a knowledge of the order of magnitude of the number of configurations of a certain type is more important than the exact number in a form which is inconvenient for calculations. Rather than report lower and upper bounds, we develop exact asymptotic numbers in Chapter 9 for several different graphical structures. Necessarily this is only illustrative, as again a whole book can be written on graphical asymptotics. Finally as a special feature we conclude with a new comprehensive definitive list of unsolved graphical enumeration problems.

The exercises range widely in difficulty from routine to intractible. Thus not all the exercises are intended to be worked out in detail by the reader. Frequently, counting formulas are given in exercises in order to include this information in the book. There are also abundantly many exercises within the text, not labeled as such, in the form of results whose proofs are omitted. We have found it convenient to indicate Equation 7 of Section 1 of Chapter 3 by the ordered triple denoted (3.1.7) and trust that the reader will forgive us for using this complicated notation. The end of a proof is marked by the symbol //.

It is our hope and belief that the present volume will make enumeration techniques more available and more unified. In turn this should serve as a stimulus for the investigation of open counting questions.

Acknowledgments

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We thank all whose names appear in the references. Helpful comments were made at various times by B. Manvel, R. C. Read, P. K. Stockmeyer, R. W. Robinson, and A. J. Schwenk. Very special thanks are due to John Riordan who gave the entire book his meticulous attention and offered many helpful suggestions. Most emphatically, each of us also thanks the other.

Finally, we thank Academic Press for their enthusiastic and effective support of graph theory and combinatorial theory. Tangible evidence of this can be found in the books cited in the bibliography and also in the existence of the first journal devoted to this fascinating subject, the *Journal of Combinatorial Theory*, founded by F. Harary and G.-C. Rota.

We offer ten cents (one U.S. dime) for each first notification of a misprint sent to either of us. Unlike Gilbert and Sullivan, we intend to continue

talking to each other. Unlike Allendoerfer and Oakley, we do not blame each other for the misprints, but we join in blaming the publisher.

Ann Arbor, Michigan FRANK HARARY

East Lansing, Michigan EDGAR M. PALMER

The Royal Mathematician was a bald-headed, nearsighted man, with a skullcap on his head and a pencil behind each ear. He wore a black suit with white numbers on it.

"I don't want to hear a long list of all the things you have figured out for me since 1907," the King said to him. "I just want you to figure out right now how to get the moon for the Princess Lenore. When she gets the moon, she will be well again."

"I am glad you mentioned all the things I have figured out for you since 1907," said the Royal Mathematician. "It so happens that I have a list of them with me."

James Thurber, *"Many Moons"*

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Don't rely too much on labels,
For too often they are fables.

C. H. Spurgeon

Chapter 1 LABELED ENUMERATION

We consider labeled enumeration problems first because they always appear to be much easier to solve than the corresponding unlabeled problems. For example, the number of labeled graphs is instantly found from first principles, while the determination of the number of unlabeled graphs requires a considerable amount of combinatorial theory including Pólya's Theorem.

We shall present in this chapter a selected sample of some of the outstanding and interesting solutions to labeled enumeration problems in graph theory, including the determination of the number of labeled graphs, connected graphs, blocks, eulerian graphs, k-colored graphs, acyclic digraphs, trees, and eulerian trails in an eulerian digraph. Often several different solutions to the same problem will be provided so that the reader has an opportunity to become acquainted with a variety of useful tricks, skills, devices, and schemes. For example, we shall see that when dealing with labeled enumeration problems, the exponential generating functions provide a natural vehicle for carrying sufficient information for a solution. On the other hand, by examining a small amount of data, one can often quickly find a required formula which can then be verified by an induction argument.

1.1 THE NUMBER OF WAYS TO LABEL A GRAPH

A *graph G of order p* consists of a finite nonempty set $V = V(G)$ of p *points* together with a specified set X of q unordered pairs of distinct points; this automatically excludes *loops* (lines joining a point to itself) and multiple lines (in parallel). A pair $x = \{u, v\}$ of points in X is called a *line* of G and x is said to *join* u and v. The points u and v are *adjacent*; u and x are *incident* with each other, as are v and x. A graph with p points and q lines is called a (p, q) *graph*. Our terminology will follow that in the book on graph theory [H1]. However, we plan to include most definitions.

It is most convenient and illuminating to represent graphs by diagrams. Consider the graph G chosen at random with

$$V = \{v_1, v_2, v_3, v_4\}$$

and

$$X = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}, \{v_1, v_3\}\}.$$

This is illustrated by the diagram in Figure 1.1.1. Only the names of the points have been used in this diagram. The five lines of G are represented by the line segments which join the pairs of points in the figure. The diagrams of all graphs of order 4, arranged by number of lines, are shown in Figure 1.1.2. Henceforth we shall also refer to such diagrams as graphs by an abuse of language which will cause no confusion.

In a *labeled graph* of order p, the integers from 1 through p are assigned to its points. For example, the random graph (of Figure 1.1.1) can be labeled in the six different ways indicated in Figure 1.1.3.

Thus two labeled graphs G_1 and G_2 are considered the same and called *isomorphic* if and only if there is a 1–1 map from $V(G_1)$ onto $V(G_2)$ which preserves not only adjacency but also the labeling. One can easily see then, that *all* of the different labelings of the random graph are displayed in Figure 1.1.3.

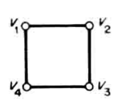

Figure 1.1.1

The graph with four points and five lines.

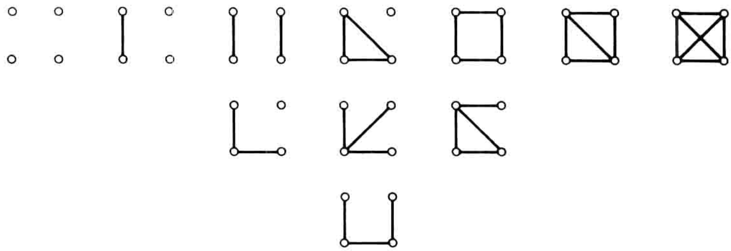

Figure 1.1.2
The 11 graphs of order 4.

Two natural questions now arise. The first asks: How many labeled graphs of order p are there? The second is: How many graphs of order p are there? The first question is so easy that we deal with it next. The second is much more difficult and will be treated in Chapter 4.

We shall answer the easier question by generalizing the problem ever so slightly to that of finding the number of labeled graphs with a given number of points *and* lines. Let $G_p(x)$ be that polynomial which has as the coefficient of x^k, the number of labeled graphs of order p which have exactly k lines. Such a polynomial is ordinarily called the "ordinary generating function" for labeled graphs with a given number of points and lines. If V is a set of p points, there are $\binom{p}{2}$ distinct unordered pairs of these points. In any labeled graph with point set V, each pair of points are either adjacent or not adjacent. The number of labeled graphs with precisely k lines is therefore $\binom{\binom{p}{2}}{k}$.

Theorem The ordinary generating function $G_p(x)$ for labeled graphs of order p is given by

$$G_p(x) = \sum_{k=0}^{m} \binom{m}{k} x^k = (1 + x)^m \tag{1.1.1}$$

where $m = \binom{p}{2}$.

Since $G_p(x) = (1 + x)^m$ and the number G_p of labeled graphs of order p is $G_p(1)$, we see that

$$G_p = 2^{\binom{p}{2}}. \tag{1.1.2}$$

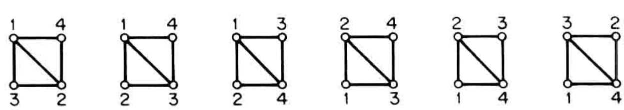

Figure 1.1.3
The six different labelings of a graph.

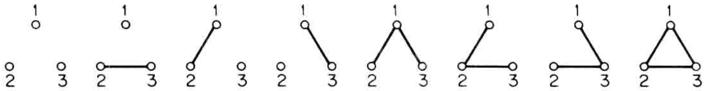

Figure 1.1.4

The eight labeled graphs of order 3.

For $p = 3$; this formula is vividly illustrated in Figure 1.1.4. Thus there are eight labeled graphs of order 3 but only four graphs of order 3; and there are 64 labeled graphs of order 4, but only 11 graphs of order 4. The question then arises: In how many ways can a given graph be labeled? To provide an answer, we must consider the symmetries or automorphisms of a graph. A 1–1 map α from $V(G)$ to $V(G_1)$ that preserves adjacency is naturally called an *isomorphism*. If $G_1 = G$, then α is an *automorphism* of G. The collection of *all* automorphisms of G, denoted $\Gamma(G)$, constitutes a group called *the group of* G. Thus the elements of $\Gamma(G)$ are *permutations* acting on V. For example, the random graph G has exactly four automorphisms, so that $\Gamma(G)$ contains the permutations in the usual cyclic representation:

$$(v_1)(v_2)(v_3)(v_4), \qquad (v_1)(v_3)(v_2v_4), \qquad (v_1v_3)(v_2)(v_4), \qquad \text{and} \qquad (v_1v_3)(v_2v_4).$$

Let $s(G) = |\Gamma(G)|$, the order of the group G, denote the number of symmetries of G. Then the answer to the labeling problem posed above is provided in the following theorem.

Theorem The number of ways of labeling a given graph G of order p is

$$l(G) = p!/s(G). \tag{1.1.3}$$

The proof is most easily obtained using some of the group theoretic results of Chapters 2 and 4, see [HPR1]. To illustrate, we simply observe that the random graph G has $p!/s(G) = 4!/4 = 6$ labelings, and the six different labeled graphs displayed in Figure 1.1.3 complete the verification of (1.1.3) for this graph G.

Although this theorem is stated only for graphs, similar versions of it hold for any finite structures with specified automorphism groups, such as rooted graphs, directed graphs, other relations of various types, simplicial complexes, functions, etc.

A *directed graph* or *digraph* D of order p consists of a finite nonempty set V of distinct objects called *points* together with a specified set X of q ordered pairs of distinct points of V. A pair $x = (u, v)$ of points in X is called an *arc* of D and u is said to be *adjacent to* v; u and x are *incident* with each other,

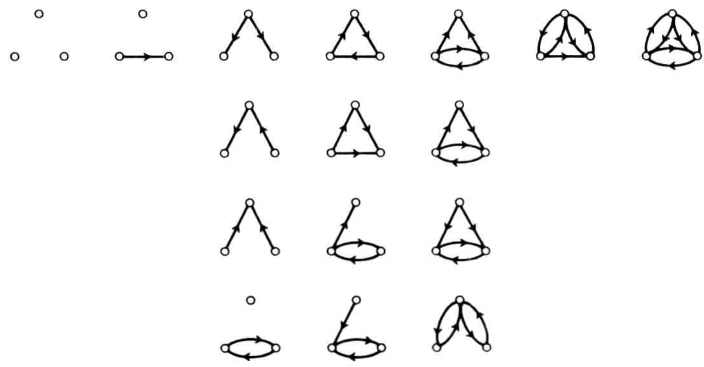

Figure 1.1.5

The 16 digraphs of order 3.

as are v and x. The *outdegree* of point u is the number of arcs with u as first point; the *indegree* as second point. The diagrams of all digraphs of order 3 are shown in Figure 1.1.5. As in the case of graphs, we refer to the diagrams themselves as digraphs.

Labeled digraphs of order p have the different integers 1 through p assigned to their points and *the group of a digraph D*, denoted $\Gamma(D)$, consists of the permutations of the points $V(D)$ of D that preserve adjacency. Since the number of labeled digraphs of order p with exactly k lines is $\binom{p(p-1)}{k}$, we have the following results which correspond to (1.1.1) and (1.1.2).

Theorem The ordinary generating function $D_p(x)$ for labeled digraphs of order p is given by

$$D_p(x) = \sum_{k=0}^{p(p-1)} \binom{p(p-1)}{k} x^k = (1 + x)^{p(p-1)}. \tag{1.1.4}$$

Obviously $D_p(x) = G_p^2(x)$ so that

$$D_p(1) = 2^{p(p-1)} = G_p^2(1). \tag{1.1.5}$$

In a round-robin tournament, a given collection of players play a game in which the rules do not allow for a draw. Any two players encounter each other just once and exactly one emerges victorious. Therefore a *tournament* is a digraph in which every pair of points are joined by exactly one arc. We conclude this section by observing that the number of labeled tournaments of order p is precisely $2^{\binom{p}{2}}$, the number, as in (1.1.2), of labeled graphs of

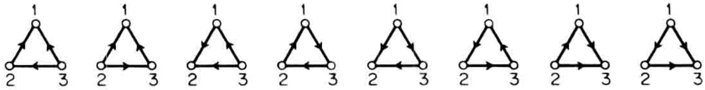

Figure 1.1.6

The eight labeled tournaments of order 3.

order p. This observation is verified for $p = 3$ by Figures 1.1.2 and 1.1.6. Furthermore, the natural correspondence between these two classes of graphs is indicated by the order in which they appear in the two figures. Each labeled tournament corresponds to that labeled graph in which the points with labels i and j are adjacent if and only if $i < j$ and the arc from i to j is present in the tournament.

1.2 CONNECTED GRAPHS

Let G be a graph and let $v_0, v_1, v_2, \ldots, v_n$ be a sequence of points of G such that v_i is adjacent to v_{i+1} for $i = 0$ to $n - 1$. Such a sequence together with these n lines, is called a *walk of length n*. If the lines $\{v_i, v_{i+1}\}$ for $i = 0$ to n are distinct, the walk is called a *trail*. If all the points are distinct (and hence the lines), it is called a *path of length n*. Then a *connected graph* is a graph in which any two points are joined by a path; see Figure 1.2.1. The number of labeled connected graphs of order 4 can be calculated by brute force if we apply (1.1.3) to each of the six graphs in Figure 1.2.1. The orders of the groups of these graphs, from left to right, are 2, 3, 2, 8, 4, 24. Then from (1.1.3) it follows that the number of labeled, connected graphs of order 4 is 38. This information provides no hint as to how to determine a formula for C_p, the number of connected, labeled graphs of order p. To that end we require the next few definitions.

A *subgraph* H of a graph G has $V(H) \subset V(G)$ and $X(H) \subset X(G)$. A *component* of a graph is a maximal, connected subgraph. A *rooted graph* has one of its points, called the *root*, distinguished from the others. Two rooted graphs are *isomorphic* if there is a 1–1 function from the point set of one graph onto that of the other which preserves not only adjacency but also the roots. A similar requirement serves to describe rooted, labeled graphs. These ideas can now be used to obtain the following recursive formula.

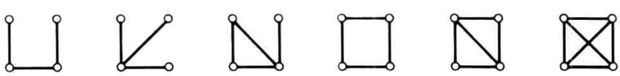

Figure 1.2.1

The six connected graphs of order 4.

Theorem The number C_p of connected, labeled graphs satisfies

$$C_p = 2^{\binom{p}{2}} - \frac{1}{p}\sum_{k=1}^{p-1} k\binom{p}{k} 2^{\binom{p-k}{2}} C_k. \tag{1.2.1}$$

To prove (1.2.1) we observe that a different rooted, labeled graph is obtained when a labeled graph is rooted at each of its points. Hence the number of rooted, labeled graphs of order p is pG_p. The number of rooted, labeled graphs in which the root is in a component of exactly k points is $kC_k\binom{p}{k}G_{p-k}$. On summing from $k = 1$ to p, we arrive again at the number of rooted, labeled graphs, namely

$$\sum_{k=1}^{p} k\binom{p}{k} C_k G_{p-k}. \qquad //$$

The values of C_p in Table 1.2.1 are listed in [S4].

TABLE 1.2.1

p	1	2	3	4	5	6	7	8	9
C_p	1	1	4	38	728	26 704	1 866 256	251 548 592	66 296 291 072

It is important to have at hand the concept of the *exponential generating function* and some of its associated properties. We shall therefore introduce these functions now and use them to provide an alternative form of (1.2.1).

For each $k = 1, 2, 3, \ldots$, let a_k be the number of ways of labeling all graphs of order k which have some property $P(a)$. Then the formal power series

$$a(x) = \sum_{k=1}^{\infty} a_k x^k/k! \tag{1.2.2}$$

is called the *exponential generating function* for the class of graphs at hand. Suppose also that

$$b(x) = \sum_{k=1}^{\infty} b_k x^k/k! \tag{1.2.3}$$

is another exponential generating function for a class of graphs with property $P(b)$.

The next lemma provides a useful interpretation of the coefficients of the product $a(x)b(x)$ of these two generating functions.