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R. J. Knops (Editor)

Nonlinear analysis and mechanics: Heriot-Watt Symposium

VOLUME I

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Preface

This volume consists mainly of written versions of invited lectures given at two short symposia held in May and September 1976 at Heriot-Watt University. The symposia formed part of a Science Research Council sponsored research programme into "Qualitative Properties of Nonlinear Elasticity", and the aim of the lectures was to describe recent methods and results in associated areas of differential equations, analysis and mechanics that might be helpful in furthering the objectives of the programme. Thus, the topics treated are not restricted to nonlinear elasticity and may be of value to those whose immediate interests do not lie in that subject.

The task of organizing the material for publication has been made easy by the thoroughness with which each author has converted the spoken to the written word. Thanks are due to them not only for this and their continual cooperation, but also for their agreement to lecture in the first place. It is also a pleasure to acknowledge the skill and patience of Mrs. Linda I. W. Gamble who accurately typed all the manuscripts, and of Pauline Houville who prepared the diagrams.

Further volumes are planned in the same series based on further lectures given as part of the S.R.C. research programme.

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R. J. Knops

Contents

By this procedure, we have made a complete analysis of the bifurcations. The procedure is more complicated than the one given in Section 1. It has the advantage of leading to generalizations to higher dimensions. This is illustrated in the next section.

10 BIFURCATION IN TWO DIMENSIONS. CUBIC NONLINEARITIES AND TWO PARAMETERS

In this section, we consider the bifurcation of solutions of the equation

$$f(u,\lambda) = 0, \qquad f \in \mathbb{R}^2,\ u \in \mathbb{R}^2,\ \lambda \in \mathbb{R}^2, \tag{10.1}$$

where for $u = (u_1, u_2)$, $\lambda = (\lambda_1, \lambda_2)$, $f(u,\lambda)$ has the form

$$f(u,\lambda) = C(u) + \lambda_1 Lu + \lambda_2 k + \text{h.o.t.}, \tag{10.2}$$

in which

$$\begin{aligned} &C: \mathbb{R}^2 \to \mathbb{R}^2 \text{ is a homogeneous cubic,} \\ &L: \text{is a } 2\times 2 \text{ matrix,} \\ &k \in \mathbb{R}^2 \text{ is given,} \\ &\text{h.o.t.} = O(|u|^4 + |\lambda_1 u|^2 + |\lambda_1|^2 + |\lambda_1\lambda_2| + |\lambda_2|^2 + |\lambda_2 u|) \end{aligned} \tag{10.3}$$

as u, λ_1, λ_2 approach zero. Our objective is to generalize the discussion in Section 9 for a scalar equation. More precisely, we will impose natural hypotheses on the function C, the matrix L and constant vector k which will permit the application of the procedure of Section 9. With these hypotheses, the complete description of the bifurcation is obtained. Details of the proofs will not be given and can be found in Chow, Hale and Mallet-Paret [17].

Our first hypothesis concerns the cubic function $C(u)$ and ensures that the cubic terms in the components of the vector C are important.

(H1) $C(u) = 0$ implies $u = 0$.

Lemma 10.1 If (H1) is satisfied, there is a neighborhood V of $(u,\lambda) = (0,0)$ and a constant $\beta \neq 0$ such that any solution of Equation (10.1) in V must satisfy

$$|u| \leq \beta (|\lambda_1|^{1/2} + |\lambda_2|^{1/3}).$$

Proof. Proceed exactly as in the proof of Lemma 9.1 to obtain a contradiction by choosing a subsequence so that $u_n/|u_n|$ approaches a constant vector γ_0 whose magnitude is obviously one and $C(\gamma_0) \neq 0$.

As for the scalar case, this justifies certain scalings in the variables. If

$$u = \lambda_2^{1/3} v, \quad \lambda_1 = \lambda_2^{2/3} \mu, \tag{10.4}$$

then the equivalent equations for v are

$$C(v) + \mu L v + k = O(|\lambda_2|^{1/3}). \tag{10.5}$$

In the scalar case, the solutions of $C(v) + k = 0$ were simple and therefore bifurcations could not occur close to the λ_2 axis in the sense of the parametrization in Relation (10.4). We certainly want the same situation here so we make the hypothesis:

(H2) If $C(v) + k = 0$ then $\det \partial C(v)/\partial v \neq 0$.

Lemma 10.2 Hypothesis (H2) implies there is a $\delta > 0$ such that all solutions of Equation (10.5) are simple for $|\mu| < \delta$, $|\lambda_2| < \delta$.

Proof. This is a direct application of the implicit function theorem.

Lemmas 10.1, 10.2 imply that it is legitimate to reparametrize a neighborhood of $\lambda = 0$ and the variable u by the relation

$$u = |\lambda_1|^{1/2} v, \quad \lambda_2 = |\lambda_1|^{3/2} \nu, \tag{10.6}$$

and only consider λ_1 small and ν in a bounded set. The resulting equivalent equations are

$$G(v,\nu,\lambda_1) \stackrel{\text{def}}{=} C(v) \pm Lv + \nu k - O(|\lambda_1|^{1/2}) \tag{10.7}$$

where ± designates the sign of λ_1 as before.

To formulate the next hypothesis, let

$$\begin{aligned} g(v,\nu) &= C(v) \pm Lv + \nu k, \\ \Delta(v) &= \det [\partial C(v)/\partial v \pm L], \\ \Delta_1(v,\nu) &= \det [\partial(g,\Delta)/\partial(v,\nu)]. \end{aligned} \tag{10.8}$$

The bifurcation curves must be determined from values of ν, λ_1 for which there are multiple solutions of Equation (10.7). In particular, for $\lambda_1 = 0$, the equations

$$g(v,\nu) = 0, \quad \Delta(v,\nu) = 0, \tag{10.9}$$

must be satisfied. These are three equations for the three unknowns (v_1, v_2, ν). Generically, these solutions should be simple (this was the case of one dimension in Section 9) and thus we impose the following hypothesis:

(H3) If $g(v,\nu) = 0$, $\Delta(v,\nu) = 0$, then $\Delta_1(v,\nu) \neq 0$.

The crucial lemma is

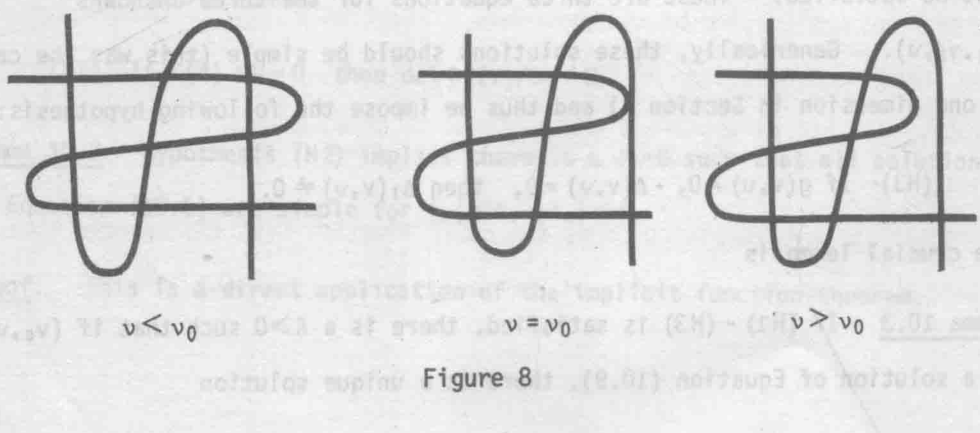
<u>Lemma 10.3</u> If (H1) - (H3) is satisfied, there is a $\delta > 0$ such that if (v_0, ν_0) is a solution of Equation (10.9), there is a unique solution

$v^*(\lambda_1)$, $\nu^*(\lambda_1)$, $|\lambda_1| < \delta$, $v^*(0) = v_0$, $\nu^*(0) = \nu_0$ of the equations

$$G(v,\nu,\lambda_1) = 0, \quad \det \partial G(v,\nu,\lambda_1)/\partial v = 0.$$

Furthermore, the curve $\lambda_2 = |\lambda_1|^{3/2}\nu^*(\lambda_1)$ is a bifurcation curve for Equation (10.1) and the number of solutions changes by exactly two as this curve is crossed. The double solution u on the bifurcation curve is given by $u = |\lambda_1|^{1/2} v^*(\lambda_1)$, $|\lambda_1| < \delta$. Finally, all bifurcation curves are obtained in this manner and they are only finite in number.

Proof. The implicit function theorem implies everything except the assertions about the fact that $\nu^*(\lambda_1)$ gives a bifurcation curve and the number of solutions changes by two as the curve is crossed. The proof of this fact is similar to the one in Section 9. Only the ideas are given and the computations may be found in [17]. For $\lambda_1 = 0$, the equation $g(u,\nu) = 0$ represents the intersection of two cubics. One can show that Hypothesis (H3) implies that when these two cubics intersect and are tangent, then the contact is of second order and, furthermore, varying the parameter ν moves the cubics apart as shown in Figure 8.



$\nu < \nu_0$ $\nu = \nu_0$ $\nu > \nu_0$

Figure 8

Lemma 10.3 gives a complete picture of the bifurcations for Equation (10.1). For a particular example, the verification of the hypotheses and the computation of the approximate bifurcation curves and solutions may be effectively accomplished on a computer. The bifurcation curves are given approximately by the cusps $\lambda_2 = |\lambda_1|^{3/2}\nu_0$ and the approximate solutions are given by $u = |\lambda_1|^{1/2}v_0$ where (v_0,ν_0) satisfy the equations (10.9).

The hypotheses given above are best possible in a certain sense which will be discussed in a later section.

11 BIFURCATION IN TWO DIMENSIONS. QUADRATIC NONLINEARITIES AND TWO PARAMETERS

The analysis in Section 10 is easily adapted to the solution of Equation (10.1) when

$$f(u,\lambda) = Q(u) + \lambda_1 Lu + \lambda_2 k + \text{h.o.t.}, \tag{11.1}$$

where L,k are the same as in (10.3) and

$$\begin{aligned} &Q: \mathbb{R}^2 \to \mathbb{R}^2 \text{ is a homogeneous quadratic,} \\ &\text{h.o.t.} = O(|u|^3 + |\lambda_1 u|^2 + |\lambda_1|^2 + |\lambda_1\lambda_2| + |\lambda_2|^2 + |\lambda_2 u|) \end{aligned} \tag{11.2}$$

as u, λ_1, $\lambda_2 \to 0$. The Hypotheses (H1) - (H3) are modified to

(H1') $Q(u) = 0$ implies $u = 0$.

With Hypothesis (H2'), the a priori bounds on the solutions are

$$|u| \leq \beta(|\lambda_1| + |\lambda_2|^{1/2}). \tag{11.3}$$

If $u = |\lambda_2|^{1/2} v$, $\lambda_1 = |\lambda_2|^{1/2}\mu$, the equivalent equations are

$$Q(v) + \mu Lv \pm k = O(|\lambda_2|^{1/2}),$$

and Hypothesis (H2) is replaced by

(H2') If $Q(v) \pm k = 0$, then $\det \partial Q(v)/\partial v \neq 0$.

With the scaling

$$u = \lambda_1 v, \quad \lambda_2 = \lambda_1^2 \nu, \tag{11.4}$$

the equivalent equations are

$$Q(v) + Lv + \nu k = O(|\lambda_1|). \tag{11.5}$$

If we define

$$h(v,\nu) = Q(v) + Lv + \nu k,$$
$$\tilde{\Delta}(v) = \det (\partial Q(v)/\partial v + L).$$
$$\tilde{\Delta}_1(v,\nu) = \det \partial(h,\tilde{\Delta})/\partial(v,\nu),$$

then Hypothesis (H3) is replaced by

(H3') If $h(v,\nu) = 0$, $\tilde{\Delta}(v) = 0$, then $\tilde{\Delta}_1(v,\nu) \neq 0$.

The conclusions are the same as before and the approximate bifurcation curves are obtained from the solutions (finite in number) of the equations

$$h(v,\nu) = 0, \quad \tilde{\Delta}(\nu) = 0, \tag{11.6}$$

and the scaling (11.4).

12 APPLICATIONS TO A SPECIAL FAMILY IN BANACH SPACE

In this section, we discuss the implications of the results in Sections 10 and 11 for the family of mappings given by Equation (5.7); that is

$$M(x,\lambda) = x-(a+\lambda_1)F(x) - \lambda_2 G(x), \tag{12.1}$$

where F,G have continuous derivatives up through order three,

$$\begin{aligned} &a \neq 0, \quad F(0) = 0, \\ &\dim \mathfrak{N}(I-aF'(0)) = 2 = \operatorname{codim} \mathcal{R}(I-aF'(0)). \end{aligned} \tag{12.2}$$

Let $w = (w_1, w_2)$ be a basis for $\mathfrak{N}(I-aF'(0))$ and $z = (z_1, z_2)$ be a basis for a complementary subspace of $\mathcal{R}(I-aF'(0))$. If $u = (u_1, u_2) \in \mathbb{R}^2$, then the bifurcation functions $f = (f_1, f_2) \in \mathbb{R}^2$ are determined from Equation (2.3) by the relation

$$\begin{aligned} z\cdot f(u,\lambda) = (I-E)[&a(F(w\cdot u + y^*(w\cdot u,\lambda)) - F'(0)y^*(w\cdot u,\lambda) \\ &+ \lambda_1 F(w\cdot u + y^*(w\cdot u,\lambda)) + \lambda_2 G(w\cdot u + y^*(w\cdot u,\lambda))], \end{aligned}$$

where $z\cdot f = z_1 f_1 + z_2 f_2$ and $w\cdot u = w_1 u_1 + w_2 u_2$.

If $f(u,\lambda)$ satisfies either relation (10.2) or (11.1), then

$$z\cdot Lu = (I-E)F'(0)w\cdot u = \frac{1}{a}(I-E)w\cdot u, \tag{12.3}$$

$$z\cdot k = (I-E)G(0), \tag{12.4}$$

$$z\cdot Q(u) = (I-E)F''(0)(w\cdot u, w\cdot u), \tag{12.5}$$

$$z\cdot C(u) = (I-E)F'''(0)(w\cdot u, w\cdot u, w\cdot u). \tag{12.6}$$

Therefore, the hypotheses (H1-H3), (H1'-H3') may be interpreted directly in terms of the functions F,G. If either of these sets of hypotheses is satisfied, then the bifurcations for the equation

$$x - (a + \lambda_1)F(x) - \lambda_2 G(x) = 0,$$

are determined by the methods of Sections 10, 11.

13 APPLICATIONS TO RECTANGULAR PLATES

In this section, we apply the results of the previous sections to the buckling of a simply supported rectangular plate subject to a lateral force and normal loading when the first eigenvalue has multiplicity two; that is, the length of one edge of the plate is $\sqrt{2}$ times the length of the other. The von Kármán equations and boundary conditions are given by Equations (6.1), (6.2) and the abstract equations are given by Equation 6.8 with $\alpha = 0$. The effect of the multiple eigenvalue on the bifurcations has been discussed by numerous authors; see, for example, Bauer, Reiss and Keller [6], Bauer and Reiss [7], Greenlee [27], Keener [41,42], Kirchgassner [44], Knightly and Sather [45], Krasnoselskii [51-53], McLeod and Sattinger [59], Matkowsky and Putnik [65], Pimbley [68] and Stakgold [78].

If $\ell = \sqrt{2}$, the first eigenvalue λ_0^{-1} of the linear operator L is $\lambda_0 = 9\pi^2/2$, $\dim \mathfrak{N}(I-\lambda_0 L) = 2 = \operatorname{codim} \mathcal{R}(I-\lambda_0 L)$ and

$$\begin{aligned} \phi_{11}(x,y) &= \frac{2^{7/4}}{3\pi^2} \sin \frac{\pi x}{\sqrt{2}} \sin \pi y, \\ \phi_{21}(x,y) &= \frac{2^{7/4}}{6\pi^2} \sin \frac{2\pi x}{\sqrt{2}} \sin \pi y, \end{aligned} \qquad (13.1)$$

are basis vectors for $\mathfrak{N}(I-\lambda_0 L)$ as well as coker $\mathcal{R}(I-\lambda_0 L)$.

If $w = u_1\phi_{11} + u_2\phi_{21} \in \mathfrak{N}(I-\lambda_0 L)$, $u = (u_1,u_2) \in \mathbb{R}^2$, then by an application of the Liapunov-Schmidt procedure and the formulas of the previous section, the bifurcation function $f = (f_1, f_2)$ and bifurcation equations are given by