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Principles of Mathematical Analysis

INTERNATIONAL SERIES IN
PURE AND APPLIED MATHEMATICS

PRINCIPLES OF MATHEMATICAL ANALYSIS

Second Edition

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INTERNATIONAL STUDENT EDITION

McGRAW-HILL BOOK COMPANY

New York San Francisco Toronto London

KŌGAKUSHA COMPANY LTD.

Tokyo

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Library of Congress Catalog Card Number 63-21479

TOSHO PRINTING CO., LTD. TOKYO, JAPAN

Preface

This book is intended to serve as a text for the course in analysis that is usually taken by advanced undergraduates or by first-year graduate students who study mathematics.

The principal difference between the present edition and the first one (published ten years ago) is that functions of several variables are now treated much more thoroughly. This change was made in response to numerous suggestions by users of the book. Chapter 9 now begins with a discussion of some basic vector-space concepts; derivatives of transformations are then defined as linear transformations; the inverse function theorem and some of its important consequences are formulated and proved in a determinant-free manner; the transformation properties of differential forms are established, and the chapter ends with a fairly general version of Stokes' theorem—the n-dimensional analogue of the fundamental theorem of calculus.

In preparation for this, Chapters 2 and 4 contain more material on Euclidean spaces, and on metric spaces, than they did. This added generality should cause no added difficulty, though. The theorems presented here are no harder in the present setting than they are on the line or in the plane.

No major changes were made in the other chapters, but much of the material was rewritten and many details (it is hoped) were improved.

The first part of Chapter 1, in which the real numbers are constructed by means of cuts in the rational number system, may be omitted at a first reading; if this is done, a logical foundation for the rest of the work can be obtained by taking the Dedekind theorem as a postulate and as a starting point. Chapters 1 to 7 should be taken up in the order in which they are presented. The three final chapters, however, are almost independent of each other.

The number of problems has been increased to about 200. Some of these involve fairly direct applications of the results obtained in the text, while others will challenge the ingenuity of the better students. Hints are supplied with most of the difficult ones.

Walter Rudin

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CHAPTER 1

The Real and Complex Number Systems

INTRODUCTION

A satisfactory discussion of the main concepts of analysis (e.g., convergence, continuity, differentiation, and integration) must be based on an accurately defined number concept. We shall not, however, enter into any discussion of the axioms governing the arithmetic of the integers, but take the rational number system as our starting point.

We shall assume familiarity with the arithmetic of the rationals (i.e., numbers of the form n/m, where n and m are integers, $m \neq 0$) and shall merely list its main features. The sum, difference, product, and quotient of any two rationals are rational (division by zero being excluded); the commutative laws

$$p + q = q + p, \qquad pq = qp,$$

the associative laws

$$(p + q) + r = p + (q + r), \qquad (pq)r = p(qr),$$

and the distributive law

$$(p + q)r = pr + qr$$

hold; and a relation $<$ is defined which introduces an order into the set of rationals. The relation $<$ has the property that for any rationals p and q we have either $p = q$ or $p < q$ or $q < p$, and it is transitive, that is, if $p < q$ and $q < r$, then $p < r$. Also, $p + q > 0$ and $pq > 0$ if $p > 0$ and $q > 0$.

It is well known that the rational number system is inadequate for many purposes. For instance, there is no rational p such that $p^2 = 2$ (we shall prove this presently). This leads to the introduction of so-called "irrational numbers," which are often written in the form of infinite decimal expansions and are considered to be approximated by the corresponding finite decimals. Thus the sequence

$$1, 1.4, 1.41, 1.414, 1.4142, \ldots$$

"tends to $\sqrt{2}$." But unless the irrational $\sqrt{2}$ has been clearly defined, the question must arise: Just what is it that the above sequence "tends to"?[1]

The main purpose of this chapter is to give the required definition.

1.1. Example. Let us begin by showing that the equation

$$p^2 = 2 \tag{1}$$

is not satisfied by any rational p. For, suppose that (1) is satisfied. Then we can write $p = m/n$, where m and n are integers, and we can further choose m and n so that not both are even. Let us assume that this is done. Then (1) implies

$$m^2 = 2n^2. \tag{2}$$

This shows that m^2 is even. Hence m is even (if m were odd, m^2 would be odd), and so m^2 is divisible by 4. It follows that the right side of (2) is divisible by 4, so that n^2 is even, which implies that n is even.

Thus the assumption that (1) holds leads us to the conclusion that both m and n are even, contrary to our choice of m and n. Hence (1) is impossible for rational p.

We now examine the situation a little more closely. Let A be the set of all positive rationals p such that $p^2 < 2$, and let B consist of all positive rationals p such that $p^2 > 2$. We shall show that A contains no largest number, and B contains no smallest.

More explicitly, for every p in A we can find a rational q in A such that $p < q$, and for every p in B we can find a rational q in B such that $q < p$.

Suppose that p is in A. Then $p^2 < 2$. Choose a rational h such that $0 < h < 1$, and such that

$$h < \frac{2 - p^2}{2p + 1}.$$

Put $q = p + h$. Then $q > p$, and

$$q^2 = p^2 + (2p + h)h < p^2 + (2p + 1)h < p^2 + (2 - p^2) = 2,$$

so that q is in A. This proves the first part of our assertion.

Next, suppose that p is in B. Then $p^2 > 2$. Put

$$q = p - \frac{p^2 - 2}{2p} = \frac{p}{2} + \frac{1}{p}.$$

[1] For a fuller discussion of this point, we refer to Knopp's "Theory and Application of Infinite Series," §1.

Then $0 < q < p$, and

$$q^2 = p^2 - (p^2 - 2) + \left(\frac{p^2 - 2}{2p}\right)^2 > p^2 - (p^2 - 2) = 2,$$

so that q is in B.

1.2. Remark. The purpose of the above discussion has been to show that the rational number system has certain gaps, in spite of the fact that between any two rationals there is another [since $p < (p + q)/2 < q$ if $p < q$]. We shall now describe a process, due to Dedekind, which fills these gaps and gives us the real numbers. For reasons of space, some details will not be carried out in full. For a complete treatment, starting with the integers, we refer to Landau's "Foundations of Analysis," which deals with the number system exclusively.

1.3. Notation. If A is any set (whose elements may be numbers, or any other objects) we write $x \in A$ to denote that x is a member (or an element) of A. If x is not a member of A, we write $x \notin A$.

The set which contains no element will be called the empty set. If a set has at least one element, it is called nonempty.

DEDEKIND CUTS

1.4. Definition. A set α of rational numbers is said to be a cut if

(I) α contains at least one rational, but not every rational;

(II) if $p \in \alpha$ and $q < p$ (q rational), then $q \in \alpha$;

(III) α contains no largest rational.

In this section, we shall always use $p, q, r, \ldots$ to denote rationals, while cuts will be denoted by $\alpha, \beta, \gamma, \ldots$ (with the exception stated in Definition 1.7).

1.5. Theorem. *If $p \in \alpha$ and $q \notin \alpha$, then $p < q$.*

Proof: If $p \in \alpha$ and $q \leq p$, (II) implies that $q \in \alpha$.

In view of this theorem, the members of α are sometimes called lower numbers of α, whereas the rationals which are not in α are called upper numbers of α. Example 1.1 shows that there need not always be a smallest upper number. However, for certain cuts, smallest upper numbers do exist:

1.6. Theorem. *Let r be rational. Let α be the set consisting of all rationals p such that $p < r$. Then α is a cut, and r is the smallest upper number of α.*

Proof: It is clear that α satisfies conditions (I) and (II) of Definition 1.4. As to (III), we need merely note that for any $p \in \alpha$,

$$p < \frac{p + r}{2} < r,$$

and therefore $(p + r)/2 \in \alpha$.

Since $r < r$ is absurd, we see that $r \notin \alpha$. Since $p < r$ implies $p \in \alpha$, r is the smallest upper number of α.

1.7. Definition. The cut constructed in Theorem 1.6 is called a rational cut. When we wish to indicate that a cut α is the rational cut related to r by the above construction, we write $\alpha = r^*$.

1.8. Definition. Let α, β be cuts. We write $\alpha = \beta$ if $p \in \alpha$ implies $p \in \beta$, and $q \in \beta$ implies $q \in \alpha$, that is, if the two sets are identical. Otherwise we write $\alpha \neq \beta$.

Note: The above definition may at first glance seem to be superfluous. But equality is not always defined as identity. For instance, if $p = a/b$ and $q = c/d$ are rational (a, b, c, d being integers), we define $p = q$ to mean $ad = bc$, but not necessarily $a = c$ and $b = d$.

We now introduce an order relation into the set of cuts.

1.9. Definition. Let α, β be cuts. We write $\alpha < \beta$ (or $\beta > \alpha$) if there is a rational p such that $p \in \beta$ and $p \notin \alpha$.

$$\alpha \leq \beta \quad \text{means} \quad \alpha = \beta \quad \text{or} \quad \alpha < \beta.$$
$$\alpha \geq \beta \quad \text{means} \quad \beta \leq \alpha.$$

If $\alpha > 0^*$, we say that α is positive; if $\alpha \geq 0^*$, we say that α is nonnegative. Similarly, if $\alpha < 0^*$, α is negative, and nonpositive if $\alpha \leq 0^*$.

We insert the remark that we shall of course continue to use the symbol $<$ between rationals, so that the symbol will (temporarily) be doing double duty. The context will always make it clear, however, which meaning is to be attached to the symbol.

1.10. Theorem. *Let α, β be cuts. Then either $\alpha = \beta$ or $\alpha < \beta$ or $\beta < \alpha$.*

Proof: Definitions 1.8 and 1.9 show clearly that if $\alpha = \beta$, neither of the other two relations can hold. To show that $\alpha < \beta$ and $\beta < \alpha$ are mutually exclusive, suppose that both these relations hold. Since $\alpha < \beta$, there is a rational p such that

$$p \in \beta, \qquad p \notin \alpha.$$

Since $\beta < \alpha$, there is a rational q such that

$$q \in \alpha, \qquad q \notin \beta.$$

By Theorem 1.5, $p \in \beta$ and $q \notin \beta$ implies $p < q$, whereas $q \in \alpha$ and $p \notin \alpha$ implies $q < p$. This is a contradiction, since $p < q$ and $q < p$ is impossible for rationals.

So far we have proved that at most one of the three relations can hold. Now suppose $\alpha \neq \beta$. Then the two sets are not identical; that is, either there is a rational p in α but not in β, in which case $\beta < \alpha$, or there is a rational q in β but not in α, in which case $\alpha < \beta$.

1.11. Theorem. *Let α, β, γ be cuts. If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.*

Proof: Since $\alpha < \beta$, there is a rational p such that

$$p \mathbin{\varepsilon} \beta, \qquad p \not\mathbin{\varepsilon} \alpha.$$

Since $\beta < \gamma$, there is a rational q such that

$$q \mathbin{\varepsilon} \gamma, \qquad q \not\mathbin{\varepsilon} \beta.$$

Now, $p \mathbin{\varepsilon} \beta$ and $q \not\mathbin{\varepsilon} \beta$ implies $p < q$; and this, together with $p \not\mathbin{\varepsilon} \alpha$, implies $q \not\mathbin{\varepsilon} \alpha$. Thus

$$q \mathbin{\varepsilon} \gamma, \qquad q \not\mathbin{\varepsilon} \alpha.$$

This means $\alpha < \gamma$.

The above two theorems show that the relation $<$ defined in Definition 1.9 does indeed have the properties which one usually associates with the concept of inequality.

We now proceed to construct an arithmetic in the set of cuts.

1.12. Theorem. *Let α, β be cuts. Let γ be the set of all rationals r such that $r = p + q$, where $p \mathbin{\varepsilon} \alpha$ and $q \mathbin{\varepsilon} \beta$. Then γ is a cut.*

Proof: We shall show that γ satisfies the three conditions of Definition 1.4.

(I) Clearly γ is not empty. Take $s \not\mathbin{\varepsilon} \alpha$, $t \not\mathbin{\varepsilon} \beta$, s and t rational. Then $s + t > p + q$ for all $p \mathbin{\varepsilon} \alpha$, $q \mathbin{\varepsilon} \beta$, so that $s + t \not\mathbin{\varepsilon} \gamma$. Hence γ does not contain every rational.

(II) Suppose $r \mathbin{\varepsilon} \gamma$, $s < r$, s rational. Then $r = p + q$ for some $p \mathbin{\varepsilon} \alpha$, $q \mathbin{\varepsilon} \beta$. Choose a rational t such that $s = t + q$. Then $t < p$; hence $t \mathbin{\varepsilon} \alpha$; hence $s \mathbin{\varepsilon} \gamma$.

(III) Suppose $r \mathbin{\varepsilon} \gamma$. Then $r = p + q$ for some $p \mathbin{\varepsilon} \alpha$, $q \mathbin{\varepsilon} \beta$. There is a rational $s > p$ such that $s \mathbin{\varepsilon} \alpha$. Hence $s + q \mathbin{\varepsilon} \gamma$ and $s + q > r$, so that r is not the largest rational in γ.

1.13. Definition. The cut γ constructed in Theorem 1.12 is denoted by $\alpha + \beta$ and is called the sum of α and β.

(The remark made after Definition 1.9 applies to the symbol $+$ as well.)

1.14. Theorem. *Let α, β, γ be cuts. Then*

(*a*) $\alpha + \beta = \beta + \alpha$;

(*b*) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$, *so that the parentheses may be omitted without ambiguity;*

(*c*) $\alpha + 0^* = \alpha$.

Proof: To construct $\alpha + \beta$, we take the set of all rationals of the form $p + q$ ($p \mathbin{\varepsilon} \alpha$, $q \mathbin{\varepsilon} \beta$). To construct $\beta + \alpha$, we take $q + p$ in place of $p + q$. By the commutative law for addition of rationals, $\alpha + \beta$ and $\beta + \alpha$ are identical cuts, which proves (*a*).

Similarly, the associative law for addition of rationals implies (*b*).

To prove (c), let $r \varepsilon \alpha + 0^*$. Then $r = p + q$ for some $p \varepsilon \alpha$, $q \varepsilon 0^*$ (that is, $q < 0$). Hence $p + q < p$, so that $p + q \varepsilon \alpha$, and $r \varepsilon \alpha$.

Next, let $r \varepsilon \alpha$. Choose $s > r$, s rational, such that $s \varepsilon \alpha$. Put $q = r - s$. Then $q < 0$, $q \varepsilon 0^*$, and $r = s + q$, so that $r \varepsilon \alpha + 0^*$.

Thus the cuts $\alpha + 0^*$ and α are identical.

1.15. Theorem. *Let α be a cut, and let $r > 0$ be a given rational. Then there are rationals p, q such that $p \varepsilon \alpha$, $q \notin \alpha$, q is not the smallest upper number of α, and $q - p = r$.*

Proof: Choose a rational $s \varepsilon \alpha$. For $n = 0, 1, 2, \ldots$, put $s_n = s + nr$. Then there is a unique integer m such that $s_m \varepsilon \alpha$ and $s_{m+1} \notin \alpha$. If s_{m+1} is not the smallest upper number of α, take $p = s_m$, $q = s_{m+1}$.

If s_{m+1} is the smallest upper number of α, take

$$p = s_m + \frac{r}{2}, \qquad q = s_{m+1} + \frac{r}{2}.$$

1.16. Theorem. *Let α be a cut. Then there is one and only one cut β such that $\alpha + \beta = 0^*$.*

Proof: We first prove uniqueness. If $\alpha + \beta_1 = \alpha + \beta_2 = 0^*$, Theorem 1.14 shows that

$$\beta_2 = 0^* + \beta_2 = (\alpha + \beta_1) + \beta_2 = (\alpha + \beta_2) + \beta_1 = 0^* + \beta_1 = \beta_1.$$

To prove existence, let β be the set of all rationals p such that $-p$ is an upper number of α, but not the smallest upper number. We have to verify that this set β satisfies the three conditions of Definition 1.4.

(I) Clear.

(II) If $p \varepsilon \beta$ and $q < p$ (q rational), then $-p \notin \alpha$, and $-q > -p$, so that $-q$ is an upper number of α, but not the smallest. Hence $q \varepsilon \beta$.

(III) If $p \varepsilon \beta$, $-p$ is an upper number of α, but not the smallest, so that there is a rational q such that $-q < -p$, and $-q \notin \alpha$. Put

$$r = \frac{p + q}{2}.$$

Then $-q < -r < -p$, so that $-r$ is an upper number of α, but not the smallest. Hence we have found a rational $r > p$ such that $r \varepsilon \beta$.

Having shown that β is a cut, we now have to verify that $\alpha + \beta = 0^*$.

Suppose $p \varepsilon \alpha + \beta$. Then $p = q + r$, for some $q \varepsilon \alpha$, $r \varepsilon \beta$. Hence $-r \notin \alpha$, $-r > q$, $q + r < 0$, and $p \varepsilon 0^*$.

Suppose $p \varepsilon 0^*$. Then $p < 0$. By Theorem 1.15, there are rationals $q \varepsilon \alpha$, $r \notin \alpha$ (and such that r is not the smallest upper number of α) such that $r - q = -p$. Since $-r \varepsilon \beta$, we have

$$p = q - r = q + (-r) \varepsilon \alpha + \beta.$$

This completes the proof.

1.17. Definition. The cut β constructed in Theorem 1.16 is denoted by $-\alpha$.

1.18. Theorem. *For any cuts α, β, γ with $\beta < \gamma$ we have $\alpha + \beta < \alpha + \gamma$. In particular (taking $\beta = 0^*$), we have $\alpha + \gamma > 0^*$ if $\alpha > 0^*$ and $\gamma > 0^*$.*

Proof: By Definitions 1.9 and 1.13, $\alpha + \beta \leq \alpha + \gamma$. If

$$\alpha + \beta = \alpha + \gamma,$$

then

$$\beta = 0^* + \beta = (-\alpha) + (\alpha + \beta) = (-\alpha) + (\alpha + \gamma) = 0^* + \gamma = \gamma,$$

by Theorem 1.14.

1.19. Theorem. *Let α, β be cuts. Then there is one and only one cut γ such that $\alpha + \gamma = \beta$.*

Proof: That there is at most one such γ follows from the fact that $\gamma_1 \neq \gamma_2$ implies $\alpha + \gamma_1 \neq \alpha + \gamma_2$ (Theorem 1.18).

Put $\gamma = \beta + (-\alpha)$. By Theorem 1.14, we have then

$$\alpha + \gamma = \alpha + [\beta + (-\alpha)] = \alpha + [(-\alpha) + \beta] = [\alpha + (-\alpha)] + \beta$$
$$= 0^* + \beta = \beta.$$

1.20. Definition. The cut γ constructed in Theorem 1.19 is denoted by $\beta - \alpha$. That is, we write $\beta - \alpha$ in place of $\beta + (-\alpha)$.

1.21. Remark. We do not require any group theory in this book. However, those readers who are familiar with the group concept may have noticed that Theorems 1.12, 1.14, and 1.16 can be summarized by saying that the set of cuts is a commutative group with respect to addition as defined by Definition 1.13. We now define multiplication, and show that a field is obtained.

Having discussed addition and subtraction of cuts in considerable detail, we shall deal briefly, and without proofs, with multiplication and division. The proofs of the theorems we shall state are quite analogous to those concerning addition and subtraction, except that it is sometimes necessary to consider several cases, depending on the signs of the factors involved.

1.22. Theorem. *Let α, β be cuts such that $\alpha \geq 0^*$, $\beta \geq 0^*$. Let γ consist of all negative rationals, plus all rationals r such that $r = pq$, where $p \,\varepsilon\, \alpha$, $q \,\varepsilon\, \beta$, $p \geq 0$, $q \geq 0$. Then γ is a cut.*

1.23. Definition. The cut constructed in Theorem 1.22 is denoted by $\alpha\beta$ and is called the product of α and β.

1.24. Definition. With every cut α we associate a cut $|\alpha|$, which we call the absolute value of α, as follows:

$$|\alpha| = \begin{cases} \alpha & \text{if } \alpha \geq 0^*, \\ -\alpha & \text{if } \alpha < 0^*. \end{cases}$$

Clearly, $|\alpha| \geq 0^*$ for all α, and $|\alpha| = 0^*$ only if $\alpha = 0^*$.

We can now complete the definition of multiplication.

1.25. Definition. Let α, β be cuts. We define

$$\alpha\beta = \begin{cases} -(|\alpha|\ |\beta|) & \text{if } \alpha < 0^*, \beta \geq 0^*, \\ -(|\alpha|\ |\beta|) & \text{if } \alpha \geq 0^*, \beta < 0^*, \\ |\alpha|\ |\beta| & \text{if } \alpha < 0^*, \beta < 0^*. \end{cases}$$

Note that the product $|\alpha|\ |\beta|$ has already been defined by Definition 1.23, since $|\alpha| \geq 0^*$, $|\beta| \geq 0^*$.

1.26. Theorem. *For any cuts α, β, γ we have*

(a) $\alpha\beta = \beta\alpha$,
(b) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$,
(c) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$,
(d) $\alpha 0^* = 0^*$,
(e) $\alpha\beta = 0^*$ *only if* $\alpha = 0^*$ *or* $\beta = 0^*$,
(f) $\alpha 1^* = \alpha$.
(g) *If* $0^* < \alpha < \beta$, *and* $\gamma > 0^*$, *then* $\alpha\gamma < \beta\gamma$.

1.27. Theorem. *If $\alpha \neq 0^*$, then for every cut β there is one and only one cut γ (which we denote by β/α) such that $\alpha\gamma = \beta$.*

We conclude this section with three theorems concerning rational cuts.

1.28. Theorem. *For any rationals p and q, we have*

(a) $p^* + q^* = (p + q)^*$,
(b) $p^*q^* = (pq)^*$,
(c) $p^* < q^*$ *if and only if* $p < q$.

Proof: If $r \,\varepsilon\, p^* + q^*$, then $r = s + t$, where $s < p$, $t < q$, so that $r < p + q$. Hence $r \,\varepsilon\, (p + q)^*$.

If $r \,\varepsilon\, (p + q)^*$, then $r < p + q$. Put

$$h = p + q - r, \qquad s = p - \frac{h}{2}, \qquad t = q - \frac{h}{2}.$$

Then $s \,\varepsilon\, p^*$, $t \,\varepsilon\, q^*$, and $r = s + t$, so that $r \,\varepsilon\, p^* + q^*$.

This proves (a). The proof of (b) is similar.

If $p < q$, then $p \,\varepsilon\, q^*$, but $p \not\varepsilon\, p^*$, so that $p^* < q^*$.

If $p^* < q^*$, there is a rational r such that $r \,\varepsilon\, q^*$, $r \not\varepsilon\, p^*$. Hence

$$p \leq r < q,$$

so that $p < q$.

1.29. Theorem. *If α, β are cuts, and $\alpha < \beta$, then there is a rational cut r^* such that $\alpha < r^* < \beta$.*

Proof: If $\alpha < \beta$, there is a rational p such that $p \,\varepsilon\, \beta$, $p \not\varepsilon\, \alpha$. Choose $r > p$ such that $r \,\varepsilon\, \beta$.

Since $r \,\varepsilon\, \beta$ and $r \not\varepsilon\, r^*$, we see that $r^* < \beta$.

Since $p \,\varepsilon\, r^*$ and $p \not\varepsilon\, \alpha$, we see that $\alpha < r^*$.

1.30. Theorem. *For any cut α, $p \varepsilon \alpha$ if and only if $p^* < \alpha$.*

Proof: For any rational p, $p \not\varepsilon p^*$. Hence $p^* < \alpha$ if $p \varepsilon \alpha$. Conversely, if $p^* < \alpha$, there is a rational q such that $q \varepsilon \alpha$ and $q \not\varepsilon p^*$. Thus $q \geq p$, which, together with $q \varepsilon \alpha$, implies $p \varepsilon \alpha$.

REAL NUMBERS

Let us now summarize the preceding section. We considered certain sets of rationals which we called cuts. An order relation and two operations, called addition and multiplication, were defined, and we proved that the resulting arithmetic of cuts obeys the same laws as the arithmetic of the rationals. In other words, the set of all cuts was made into an ordered field.

A special class of cuts, the so-called "rational cuts," was singled out for special attention, and we found that the replacement of the rational numbers r by the corresponding cuts r^* preserves sums, products, and order (Theorem 1.28). This fact may be expressed by saying that the ordered field of all rational numbers is *isomorphic* to the ordered field of all rational cuts, and it enables us to identify the rational cut r^* with the rational number r. Of course, r^* is by no means the same as r, but the properties we are concerned with (arithmetic and order) are the same in the two fields.

We now define what we mean by a real number.

1.31. Definition. Cuts will from now on be called real numbers. Rational cuts will be identified with rational numbers (and will be called rational numbers). All other cuts will be called irrational numbers.

We thus obtain the rationals as a subset of the real number system. Theorem 1.29 shows that between any two reals there is a rational, and Theorem 1.30 shows that every real number α is the set of all rationals p such that $p < \alpha$.

The following theorem states a very fundamental property of the real number system.

1.32. Theorem (Dedekind). *Let A and B be sets of real numbers such that*

(*a*) *every real number is either in A or in B;*

(*b*) *no real number is in A and in B;*

(*c*) *neither A nor B is empty;*

(*d*) *if $\alpha \varepsilon A$, and $\beta \varepsilon B$, then $\alpha < \beta$.*

Then there is one (and only one) real number γ such that $\alpha \leq \gamma$ for all $\alpha \varepsilon A$, and $\gamma \leq \beta$ for all $\beta \varepsilon B$.

Before giving the proof, let us state the following Corollary.

Corollary. *Under the hypotheses of Theorem* 1.32, *either A contains a largest number or B contains a smallest.*

For if $\gamma \epsilon A$, γ is the largest number in A; if $\gamma \epsilon B$, γ is the smallest number in B; and by (*a*) one of these two cases must occur, while (*b*) implies that both cannot occur together.

It is the existence of γ (uniqueness is trivial) which is the important feature of the theorem and which shows that the gaps which we found in the rational number system (compare Example 1.1) are now filled. Moreover, if we tried to repeat the process which led us from the rationals to the reals, by constructing cuts (as in Definition 1.4) whose members are real numbers, every cut would have a smallest upper number, we could immediately identify every cut with its smallest upper number, and nothing new would be obtained.

For this reason, Theorem 1.32 is sometimes called the completeness theorem for the real numbers.

Proof of Theorem 1.32: Suppose there are two numbers, γ_1 and γ_2, for which the conclusion holds, and $\gamma_1 < \gamma_2$. Choose γ_3 such that $\gamma_1 < \gamma_3 < \gamma_2$ (this is possible, by Theorem 1.29). Then $\gamma_3 < \gamma_2$ implies $\gamma_3 \epsilon A$, whereas $\gamma_1 < \gamma_3$ implies $\gamma_3 \epsilon B$. This contradicts (*b*). There is thus at most one number γ with the desired properties.

We let γ be the set of all rational p such that $p \epsilon \alpha$ for some $\alpha \epsilon A$. We have to verify that γ satisfies the conditions of Definition 1.4.

(I) Since A is not empty, neither is γ. If $\beta \epsilon B$ and $q \not\epsilon \beta$, then $q \not\epsilon \alpha$ for any $\alpha \epsilon A$ (since $\alpha < \beta$); hence $q \not\epsilon \gamma$.

(II) If $p \epsilon \gamma$ and $q < p$, then $p \epsilon \alpha$ for some $\alpha \epsilon A$; hence $q \epsilon \alpha$; hence $q \epsilon \gamma$.

(III) If $p \epsilon \gamma$, then $p \epsilon \alpha$ for some $\alpha \epsilon A$; hence there exists $q > p$ such that $q \epsilon \alpha$; hence $q \epsilon \gamma$.

Thus γ is a real number.

It is clear that $\alpha \leq \gamma$ for all $\alpha \epsilon A$. If there were some $\beta \epsilon B$ for which $\beta < \gamma$, then there would be a rational p such that $p \epsilon \gamma$, and $p \not\epsilon \beta$; but if $p \epsilon \gamma$, then $p \epsilon \alpha$ for some $\alpha \epsilon A$, and this implies that $\beta < \alpha$, a contradiction to (*d*). Thus $\gamma \leq \beta$ for all $\beta \epsilon B$, and the proof is complete.

We shall now discard some of the notational conventions used so far: The letters $p, q, r, \ldots$ will no longer be reserved for rationals, and $\alpha, \beta, \gamma, \ldots$ will also be available for general use.

1.33. Definition. Let E be a set of real numbers. If there is a number y such that $x \leq y$ for all $x \epsilon E$, we say that E is bounded above, and call y an upper bound of E.

Lower bounds are defined in the same way. If E is bounded above and below, then E is said to be bounded.

1.34. Definition. Let E be bounded above. Suppose y has the following properties:

(*a*) y is an upper bound of E;

(*b*) *if* $x < y$, then x is not an upper bound of E.