Australian Mathematical Society Lecture Series 22

Representations of Lie Algebras

An Introduction Through $\mathfrak{gl}_n$

Anthony Henderson

CAMBRIDGE

Australian Mathematical Society Lecture Series: 22

Representations of Lie Algebras

An Introduction Through $\mathfrak{gl}_n$

ANTHONY HENDERSON

School of Mathematics and Statistics
University of Sydney

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press,
New York

www.cambridge.org
Information on this title: www.cambridge.org/9781107653610

First published 2012

Printed and bound in the United Kingdom by the MPG Books Group

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Henderson, Anthony, 1976- author.
Representations of Lie algebras : an introduction through gln / Anthony Henderson, School of
Mathematics and Statistics, University of Sydney.
pages cm. – (Australian Mathematical Society lecture series ; 22)
ISBN 978-1-107-65361-0 (pbk.)
1. Representations of Lie algebras. I. Title.
QA252.3.H46 2012
512′.482–dc23
2012021841

ISBN 978-1-107-65361-0 Paperback

Representations of Lie Algebras

This bold and refreshing approach to Lie algebras assumes only modest prerequisites (linear algebra up to the Jordan canonical form and a basic familiarity with groups and rings), yet it reaches a major result in representation theory: the highest-weight classification of irreducible modules of the general linear Lie algebra. The author's exposition is focused on this goal rather than on aiming at the widest generality, and emphasis is placed on explicit calculations with bases and matrices. The book begins with a motivating chapter explaining the context and relevance of Lie algebras and their representations and concludes with a guide to further reading. Numerous examples and exercises with full solutions are included.

Based on the author's own introductory course on Lie algebras, this book has been thoroughly road-tested by advanced undergraduate and beginning graduate students and is also suited to individual readers wanting an introduction to this important area of mathematics.

AUSTRALIAN MATHEMATICAL SOCIETY LECTURE SERIES

1 Introduction to Linear and Convex Programming, N. CAMERON
2 Manifolds and Mechanics, A. JONES, A. GRAY & R. HUTTON
3 Introduction to the Analysis of Metric Spaces, J. R. GILES
4 An Introduction to Mathematical Physiology and Biology, J. MAZUMDAR
5 2-Knots and their Groups, J. HILLMAN
6 The Mathematics of Projectiles in Sport, N. DE MESTRE
7 The Petersen Graph, D. A. HOLTON & J. SHEEHAN
8 Low Rank Representations and Graphs for Sporadic Groups, C. E. PRAEGER & L. H. SOICHER
9 Algebraic Groups and Lie Groups, G. I. LEHRER (ed.)
10 Modelling with Differential and Difference Equations, G. FULFORD, P. FORRESTER & A. JONES
11 Geometric Analysis and Lie Theory in Mathematics and Physics, A. L. CAREY & M. K. MURRAY (eds.)
12 Foundations of Convex Geometry, W. A. COPPEL
13 Introduction to the Analysis of Normed Linear Spaces, J. R. GILES
14 Integral: An Easy Approach after Kurzweil and Henstock, L. P. YEE & R. VYBORNY
15 Geometric Approaches to Differential Equations, P. J. VASSILIOU & I. G. LISLE (eds.)
16 Industrial Mathematics, G. R. FULFORD & P. BROADBRIDGE
17 A Course in Modern Analysis and its Applications, G. COHEN
18 Chaos: A Mathematical Introduction, J. BANKS, V. DRAGAN & A. JONES
19 Quantum Groups, R. STREET
20 Unitary Reflection Groups, G. I. LEHRER & D. E. TAYLOR
21 Lectures on Real Analysis, F. LÁRUSSON

Preface

The aim of this book

Why another introduction to Lie algebras? The subject of this book is one of the areas of algebra that has been most written about. The basic theory was unearthed more than a century ago and has been polished in a long chain of textbooks to a sheen of classical perfection. Experts' shelves are graced by the three volumes of Bourbaki [1]; for students with the right background and motivation to learn from them, the expositions in the books by Humphreys [10], Fulton and Harris [6], and Carter [2] could hardly be bettered; and there is a recent undergraduate-level introduction by Erdmann and Wildon [4]. So where is the need for this book?

The answer comes from my own experience in teaching courses on Lie algebras to Australian honours-level undergraduates (see the Acknowledgements section). Such courses typically consist of 24 one-hour lectures. At my own university the algebraic background knowledge of the students would be: linear algebra up to the Jordan canonical form, the basic theory of groups and rings, the rudiments of group representation theory, and a little multilinear algebra in the context of differential forms. From that starting point, I have found it difficult to reach any peak of the theory by following the conventional route. My definition of a peak includes the classification of simple Lie algebras, the highest-weight classification of their modules, and the combinatorics of characters, tensor products, and crystal bases; by 'the conventional route' I mean the path signposted by the theorems of Engel and Lie (about solvability), Cartan (about the Killing form), Weyl (about complete reducibility), and Serre, as in the book by Humphreys [10]. Following that path without skipping proofs always seemed to require more than 24 lectures.

The solution adopted in this book is drastic. I have abandoned the wider class of simple Lie algebras, focusing instead on the general linear Lie algebra $\mathfrak{gl}_n$, which is almost, but not quite, simple. I have jettisoned all five of the aforementioned theorems, in favour of arguments specific to $\mathfrak{gl}_n$, especially the use of explicit Casimir operators. Although these omissions may shock the experts, I have found this to be an approach that is more accessible and yet still reaches one peak: the classification of $\mathfrak{gl}_n$-modules by their highest weights.

I have started the journey with a motivatory chapter, which gives some explanation of why algebraists care about this classification and also introduces some necessary multilinear algebra. Chapters 2 to 4 cover the basic definitions of Lie algebras, homomorphisms and isomorphisms, subalgebras, ideals, quotients, modules, irreducibility and complete reducibility. In a lecture course, the material in these first four chapters would typically take about 12 hours; so the elegant $\mathfrak{sl}_2$ theory in Chapter 5 is reached relatively early. Then in Chapter 6 I return to the theory of modules, covering tensor products, bilinear forms, Schur's lemma, and Casimir operators.

In Chapter 7 these tools are used to develop the highest-weight theory. My hope is that students who reach the end of Chapter 7 will be inspired to progress to more comprehensive books, and Chapter 8 is intended as a map of what lies ahead.

Acknowledgements

This book began life as a set of lecture notes for my Lie algebras course in the 2004 Australian Mathematical Sciences Institute (AMSI) Summer School. It was extensively revised over the next seven years, as I taught the subject again for the summer school and as an honours course at the University of Sydney. Most of the exercises were originally assignment or exam questions.

I would like to thank AMSI for the initial opportunity to teach this beautiful subject, and the students in all those classes for their feedback. I would also like to thank Pramod Achar, Wai Ling Yee, Cheryl Praeger, and the anonymous reviewers for their valuable suggestions and encouraging comments.

Notational conventions

To simplify matters, we make a standing convention:

All vector spaces are over $\mathbb{C}$ and finite-dimensional.

The finite-dimensionality assumption allows the explicit calculations with bases and matrices that are a feature of the book. The $n \times n$ identity matrix is written 1_n, and the identity transformation of a vector space V is written 1_V. The elements of the vector space $\mathbb{C}^n$ are always thought of as column vectors; linear transformations of this particular vector space are tacitly identified with $n \times n$ matrices (multiplying on the left of the vectors). The bases of vector spaces are considered to be ordered sets and hence are written without set braces. The span of the elements $v_1, \ldots, v_k$ is written $\mathbb{C}\{v_1, \ldots, v_k\}$. The term 'subspace' always means 'sub-vector-space'. If W and W' are subspaces of a larger vector space V then $W \oplus W'$ denotes their sum, and it is implied that $W \cap W' = \{0\}$ (an 'internal direct sum'); if W and W' are not subspaces of a larger vector space V then $W \oplus W'$ means the 'external direct sum' $\{(w, w') \mid w \in W, w' \in W'\}$. The same principles apply to direct sums with more than two summands.

On its rare appearances, the square root of -1 is written $\mathbf{i}$ to distinguish it from the italic letter i, which is widely used for other purposes. The group of nonzero complex numbers is written $\mathbb{C}^\times$. The set of nonnegative integers is written $\mathbb{N}$. Other notation will be explained as it is needed.

Contents

CHAPTER 1

Motivation: representations of Lie groups

Sophus Lie was a Norwegian mathematician who lived from 1842 to 1899. Essentially single-handedly he discovered two fundamental classes of objects in modern mathematics, which now bear his name: Lie groups and Lie algebras. More importantly, he built a bridge between them; this is remarkable, because Lie groups seem to be part of differential geometry (in today's language) while Lie algebras seem to be purely algebraic. In this chapter we will discuss a small part of Lie's discovery.

1.1 Homomorphisms of general linear groups

Typically, Lie groups are infinite groups whose elements are invertible matrices with real or complex entries. So they are subgroups of the *general linear group*

$$GL_n = \{g \in \mathrm{Mat}_n \mid \det(g) \neq 0\},$$

where $\mathrm{Mat}_n = \mathrm{Mat}_n(\mathbb{C})$ denotes the set of $n \times n$ complex matrices for some positive integer n. Lie was interested in such groups because they give the symmetries of differential equations, but they have since found many other applications in areas such as differential geometry and harmonic analysis.

One of the most important algebraic problems concerning Lie groups is to classify a suitable class of *matrix representations* of a given Lie group G, i.e. group homomorphisms $G \to GL_m$ for various m. For the purposes of motivation, we concentrate on the case where G is the full general linear group GL_n; thus the problem can be stated (vaguely) as follows.

Problem 1.1.1. Describe all group homomorphisms $\Phi : GL_n \to GL_m$.

By definition, such a homomorphism is a map $\Phi : GL_n \to \mathrm{Mat}_m$ such that:

$$\Phi(1_n) = 1_m, \tag{1.1.1}$$

where 1_n denotes the $n \times n$ identity matrix, and

$$\Phi(gh) = \Phi(g)\Phi(h) \quad \text{for all } g, h \in GL_n. \tag{1.1.2}$$

(The case $h = g^{-1}$ of (1.1.2), combined with (1.1.1), forces $\Phi(g)$ to be invertible.) Such a map Φ is a collection of m^2 functions $\Phi_{ij} : GL_n \to \mathbb{C}$, where $\Phi_{ij}(g)$ is the (i, j) entry of the matrix $\Phi(g)$. Each function Φ_{ij} is in effect a function of n^2 variables, the entries of the input matrix g (the given domain consists of just the invertible matrices, so the function may or may not be defined for those choices of variables that give a zero determinant). So (1.1.1) and (1.1.2) amount to a complicated system of functional equations. To frame Problem 1.1.1 rigorously, we would have to specify what kinds of function are allowed as solutions – for example, continuous, differentiable, rational, or polynomial – but we will leave this undetermined for now and see what happens in some examples.

Example 1.1.2. The determinant $\det : GL_n \to \mathbb{C}^\times$ is one such homomorphism, if we make the obvious identification of $\mathbb{C}^\times$ with GL_1. The determinant of a matrix is clearly a polynomial function of the entries. ■

Example 1.1.3. The transpose map $GL_n \to GL_n : g \mapsto g^{\mathsf{t}}$ is not an example because it is an anti-automorphism rather than an automorphism: $(gh)^{\mathsf{t}}$ equals $h^{\mathsf{t}}g^{\mathsf{t}}$ and doesn't usually equal $g^{\mathsf{t}}h^{\mathsf{t}}$. But this means that the map $GL_n \to GL_n : g \mapsto (g^{\mathsf{t}})^{-1}$ is an example. The entries of $(g^{\mathsf{t}})^{-1}$ are rational functions of the entries of g: they are quotients of various $(n-1) \times (n-1)$ minors and the determinant. Therefore these functions are not defined on matrices with zero determinant. ■

Example 1.1.4. A map that is easily seen to satisfy (1.1.1) and (1.1.2) is the 'duplication' map

$$GL_2 \to GL_4 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

To produce something less trivial-looking, we could replace either copy of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with its conjugate $X\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)X^{-1}$, for some fixed $X \in GL_2$, or indeed we could conjugate the whole output matrix by some fixed $Y \in GL_4$. This is a superficial change that we could account for by introducing a suitable equivalence relation into the statement of Problem 1.1.1. Note that in this example the entries of the output matrix are linear functions of the entries of the input matrix. ■

Example 1.1.5. More interesting is the map $\Psi : GL_2 \to GL_3$ defined by

$$\Psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix},$$

where the entries of the output are homogeneous polynomials of degree 2 in the entries of the input. It is clear that property (1.1.1) is satisfied. The proof of property (1.1.2) is as follows:

$$\Psi\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Psi\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}\begin{pmatrix} e^2 & 2ef & f^2 \\ eg & eh+fg & fh \\ g^2 & 2gh & h^2 \end{pmatrix}$$

$$= \begin{pmatrix} (ae+bg)^2 & 2(ae+bg)(af+bh) & (af+bh)^2 \\ (ae+bg)(ce+dg) & \begin{matrix}(ae+bg)(cf+dh)\\ +(af+bh)(ce+dg)\end{matrix} & (af+bh)(cf+dh) \\ (ce+dg)^2 & 2(ce+dg)(cf+dh) & (cf+dh)^2 \end{pmatrix}$$

$$= \Psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right).$$

At the moment this seems like an accident, and it is not clear how to find other such solutions of (1.1.1) and (1.1.2). ■

1.2 Multilinear algebra

The right context for explaining the above examples of homomorphisms, and for finding new examples, is the theory of multilinear algebra. If V is an n-dimensional vector space with chosen basis $v_1, \dots, v_n$ then the elements of GL_n correspond bijectively to invertible linear transformations of V: a matrix (a_{ij}) in GL_n corresponds to the unique linear map $\tau : V \to V$ such that

$$\tau(v_j) = \sum_{i=1}^{n} a_{ij} v_i \quad \text{for all } j. \tag{1.2.1}$$

If we have a way of constructing from V a new vector space W with basis $w_1, \dots, w_m$, and if this construction is sufficiently 'natural', then each linear transformation of V should induce a linear transformation of W and the resulting map $\Phi : GL_n \to GL_m$ of matrices should satisfy (1.1.1) and (1.1.2). This is one reason to be interested in Problem 1.1.1: the homomorphisms between general linear groups tell us something about natural constructions of vector spaces.

Example 1.2.1. A very important example of such a homomorphism occurs when W is the *dual space* V^*, consisting of all linear functions $f : V \to \mathbb{C}$. This is also n-dimensional: it has a basis $v_1^*, \dots, v_n^*$, where v_i^* is the unique linear function satisfying

$$v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.2}$$

In other words, v_i^* is the function whose value on $a_1v_1 + \cdots + a_nv_n \in V$ is the coefficient a_i. A general linear function $f : V \to \mathbb{C}$ can be written as $f(v_1)v_1^* + \cdots + f(v_n)v_n^*$. If τ is an invertible linear transformation of V then τ induces in a natural way an invertible linear transformation τ^* of V^*, defined by

$$\tau^*(f)(v) = f(\tau^{-1}(v)) \quad \text{for all } v \in V, f \in V^*. \tag{1.2.3}$$

(The transformation τ^{-1} on the right-hand side does indeed give the function that one would naturally expect, for the same reason that, in calculus, translating the graph of $y = f(x)$ one unit to the right gives the graph of $y = f(x-1)$.) To find the matrix of τ^* relative to the basis $v_1^*, \ldots, v_n^*$, observe that its (j,i) entry is the coefficient of v_j^* in $\tau^*(v_i^*)$; this is the same as $\tau^*(v_i^*)(v_j) = v_i^*(\tau^{-1}(v_j))$, the coefficient of v_i in $\tau^{-1}(v_j)$, i.e. the (i,j) entry of the matrix of τ^{-1} relative to $v_1, \ldots, v_n$. So, the map of matrices corresponding to $\tau \mapsto \tau^*$ is the inverse transpose map considered in Example 1.1.3. ■

Example 1.2.2. Take $W = V \oplus V = \{(v, v') \mid v, v' \in V\}$. Any linear transformation τ of V induces a linear transformation $\tau \oplus \tau$ of $V \oplus V$, defined by

$$(\tau \oplus \tau)(v, v') = (\tau(v), \tau(v')) \quad \text{for all } v, v' \in V. \tag{1.2.4}$$

The most obvious basis for $V \oplus V$ consists of

$$(v_1, 0), (v_2, 0), \ldots, (v_n, 0), (0, v_1), (0, v_2), \ldots, (0, v_n).$$

Relative to this basis, the matrix corresponding to $\tau \oplus \tau$ is exactly the block-diagonal duplication of the matrix of τ seen in Example 1.1.4; the conjugated versions mentioned there would arise if one used other bases of $V \oplus V$. ■

To explain Examples 1.1.2 and 1.1.5 similarly, we need the concept of the *tensor product*, which for finite-dimensional vector spaces can be explained fairly simply. Given two vector spaces V and W with respective bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$, the tensor product $V \otimes W$ is a vector space with basis $v_i \otimes w_j$ for all i, j with $1 \le i \le n$, $1 \le j \le m$. One can regard the elements $v_i \otimes w_j$ merely as symbols and $V \otimes W$ as the space of formal linear combinations of them. Note that the dimension of $V \otimes W$ is $(\dim V)(\dim W)$, in contrast with that of the direct sum $V \oplus W$, which is $\dim V + \dim W$. For arbitrary elements $v \in V$ and $w \in W$, we define the *pure tensor* $v \otimes w \in V \otimes W$ by the following rule:

$$\begin{gathered} \text{if } v = a_1v_1 + \cdots + a_nv_n \quad \text{and} \quad w = b_1w_1 + \cdots + b_mw_m \\ \text{then } v \otimes w = \sum_{i=1}^{n} \sum_{j=1}^{m} a_ib_j(v_i \otimes w_j). \end{gathered} \tag{1.2.5}$$

Note that if v happens to equal v_i and w happens to equal w_j then $v \otimes w$ does indeed equal the basis element $v_i \otimes w_j$, so our notation is consistent. Having made this definition, one can easily show that the tensor product does not depend on the chosen bases of V and W: for any other bases $v'_1, \ldots, v'_n$ and $w'_1, \ldots, w'_m$ the elements $v'_i \otimes w'_j$ form another, equally good, basis of $V \otimes W$. It is important to bear in mind that a general element of $V \otimes W$ is not a pure tensor: it is, of course, a linear combination of the basis elements $v_i \otimes w_j$ but the coefficients cannot usually be written in the form $a_i b_j$, as in (1.2.5).

So, we have another way to construct a new vector space from a vector space V: we can consider its tensor square $V^{\otimes 2} = V \otimes V$. Any linear transformation τ of V induces a linear transformation $\tau \otimes \tau$ of $V \otimes V$, defined on the basis elements by $(\tau \otimes \tau)(v_i \otimes v_j) = \tau(v_i) \otimes \tau(v_j)$. It is easy to see that in fact

$$(\tau \otimes \tau)(v \otimes v') = \tau(v) \otimes \tau(v') \quad \text{for any } v, v' \in V. \tag{1.2.6}$$

Example 1.2.3. Suppose that V is two-dimensional, with basis v_1, v_2. If the linear transformation $\tau : V \to V$ has matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ relative to this basis then, for instance,

$$\begin{aligned}(\tau \otimes \tau)(v_1 \otimes v_1) &= \tau(v_1) \otimes \tau(v_1) \\ &= (av_1 + cv_2) \otimes (av_1 + cv_2) \\ &= a^2(v_1 \otimes v_1) + ac(v_1 \otimes v_2) + ac(v_2 \otimes v_1) + c^2(v_2 \otimes v_2).\end{aligned}$$

This calculation gives the first column of the matrix of $\tau \otimes \tau$ relative to the basis $v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2$. The whole matrix is

$$\begin{pmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{pmatrix}.$$

The map sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to this matrix is a homomorphism from GL_2 to GL_4. To check (1.1.2) there is no need to make explicit matrix multiplications as in Example 1.1.5: the relation (1.1.2) follows from the fact that, for any linear transformations τ, τ' of V,

$$(\tau \circ \tau') \otimes (\tau \circ \tau') = (\tau \otimes \tau) \circ (\tau' \otimes \tau'), \tag{1.2.7}$$

which in turn follows because the two sides take the same values when evaluated on the basis elements. ■

As can be seen in Example 1.2.3, there are two subspaces of $V \otimes V$ that are guaranteed to be preserved by all linear transformations of the form $\tau \otimes \tau$: these are the space of *symmetric tensors*, $\mathrm{Sym}^2(V)$, consisting of elements that are invariant under the interchange map $v_i \otimes v_j \mapsto v_j \otimes v_i$, and the space of *alternating tensors*, $\mathrm{Alt}^2(V)$, consisting of elements that change sign under this interchange map. By restricting $\tau \otimes \tau$ to these subspaces we obtain further homomorphisms of the type referred to in Problem 1.1.1.

Example 1.2.4. Continuing with V two-dimensional, as in Example 1.2.3, $\mathrm{Sym}^2(V)$ is three-dimensional with basis $v_1 \otimes v_1, v_1 \otimes v_2 + v_2 \otimes v_1, v_2 \otimes v_2$. The resulting homomorphism is exactly the map $\Psi : GL_2 \to GL_3$ of Example 1.1.5. By contrast, $\mathrm{Alt}^2(V)$ is one-dimensional, spanned by $v_1 \otimes v_2 - v_2 \otimes v_1$. If $\tau : V \to V$ has matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then

$$\begin{aligned}(\tau \otimes \tau)(v_1 \otimes v_2 - v_2 \otimes v_1) &= (av_1 + cv_2) \otimes (bv_1 + dv_2) \\ &\quad - (bv_1 + dv_2) \otimes (av_1 + cv_2) \\ &= (ad - bc)(v_1 \otimes v_2 - v_2 \otimes v_1).\end{aligned}$$

So the resulting homomorphism is the determinant $\det : GL_2 \to GL_1$, as in Example 1.1.2. ■

In general, if V has a basis $v_1, \ldots, v_n$ then an element of $V \otimes V$ lies in $\mathrm{Sym}^2(V)$ if and only if the coefficient of $v_i \otimes v_j$ equals the coefficient of $v_j \otimes v_i$ for all i, j. Hence $\mathrm{Sym}^2(V)$ has a basis consisting of the following elements:

$$v_i \otimes v_i \text{ for } 1 \le i \le n \quad \text{and} \quad v_i \otimes v_j + v_j \otimes v_i \qquad \text{for } 1 \le i < j \le n.$$

An element of $V \otimes V$ lies in $\mathrm{Alt}^2(V)$ if and only if the coefficient of $v_i \otimes v_i$ is zero for every i and the coefficient of $v_i \otimes v_j$ is the negative of the coefficient of $v_j \otimes v_i$ for all $i \neq j$. Hence $\mathrm{Alt}^2(V)$ has a basis consisting of the elements

$$v_i \otimes v_j - v_j \otimes v_i \quad \text{for } 1 \le i < j \le n.$$

Clearly we have a direct sum decomposition,

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V), \tag{1.2.8}$$

and the dimensions of $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ are $\binom{n+1}{2}$ and $\binom{n}{2}$ respectively.

As well as tensor squares, one can define higher tensor powers in an entirely analogous way: $V^{\otimes 3} = V \otimes V \otimes V$, $V^{\otimes 4} = V \otimes V \otimes V \otimes V$, and so forth. If V has a basis $v_1, \ldots, v_n$ then the k-fold tensor power $V^{\otimes k}$ has a basis consisting of the pure tensors:

$$v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \quad \text{for } 1 \le i_1, \ldots, i_k \le n.$$

So $\dim V^{\otimes k} = n^k$. (By convention, $V^{\otimes 1}$ is V itself.) The space of symmetric tensors $\mathrm{Sym}^k(V)$ consists of those elements of $V^{\otimes k}$ that are fixed under any permutation of the tensor factors. In other words, the coefficients of $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$ and $v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_k}$ have to be the same whenever $j_1, \ldots, j_k$ can be obtained by rearranging $i_1, \ldots, i_k$. So $\mathrm{Sym}^k(V)$ has a basis consisting of all the elements

$$t_{(k_1,\ldots,k_n)} := \sum_{\substack{1 \le s_1, \ldots, s_k \le n, \\ k_i \text{ of the } s_j \\ \text{equal } i}} v_{s_1} \otimes \cdots \otimes v_{s_k},$$