Generalized Functions, Volume 3

Theory of Differential Equations

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Translator's Note

According to the wish of Professor Gel'fand, this translation has been compared with the 1964 German translation,[1] and all improvements and omissions contained in the latter were taken over here. Certain minor corrections were made without being mentioned and a few notes were added (which are identified as translator's notes), especially in the last chapter, which is closely related to Chapter I in Volume 4 of this series.

No serious attempt has been made to coordinate the terminology with that used in previously published volumes, partly because the present translator does not entirely agree with it (e.g. the use of conjugate space for what is called here dual space, or function of bounded support, for what is more frequently called function of compact support). On the other hand, there are no radical departures from the notation and terminology of the authors—in particular, no attempt has been made to "modernize" it.[2]

The theory of partial differential equations, and of generalized eigenfunction expansions has made tremendous progress in the past few years. Not being a specialist in these fields the translator has made no attempt to update the literature on the subject (except for a few obvious references).

It was the express wish of Professor Gel'fand to refer the reader to the "excellent book of Hörmander" for some of the more recent developments.[3] This book is indeed the most valuable contribution to the literature on partial differential equations and should be read by any serious student of the subject.

Finally, I would like to thank Professor Gel'fand for supplying me with a copy of the German edition of this book and other literature which was useful in the translation.

October, 1967 MEINHARD E. MAYER

[1] "Verallgemeinerte Funktionen (Distributionen). Volume III—Einige Fragen zur Theorie der Differentialgleichungen." VEB-Deutscher Verlag der Wissenschaften, Berlin, 1964.

[2] As was done in the recently published French translation, *Math. Rev.* 1080 (1966), rev. Nr. 6001.

[3] L. Hörmander, "Linear Partial Differential Operators." Springer-Verlag, Berlin-Heidelberg-Göttingen and Academic Press, New York, 1963.

Preface to the Russian Edition

In the present volume, the third in the series "Generalized Functions," the apparatus of generalized functions is applied to the investigation of the following problems of the theory of partial differential equations: the problems of determining uniqueness and correctness classes for solutions of the Cauchy problem for systems with constant (or only time-dependent) coefficients and the problem of eigenfunction expansions for self-adjoint differential operators.

In subsequent volumes, the authors intend to discuss boundary value problems for elliptic equations and the Cauchy problem for equations with variable coefficients and for quasilinear equations, as well as problems related to complex extensions of all independent variables.

The authors use this occasion to thank the participants of the Seminar on Generalized Functions and Partial Differential Equations at Moscow State University, where various sections of this volume were repeatedly discussed. In particular, they are grateful to V. M. Borok, A. G. Kostyuchenko, Ya. I. Zhitomirskii and G. N. Zolotarev. The authors would also like to thank I. I. Shulishova for setting up detailed indexes for the first three volumes and to M. S. Agranovich, who has carefully edited the whole text and whose criticism has contributed considerable improvements.

Moscow, 1958

I. M. GEL'FAND
G. E. SHILOV

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CHAPTER I

SPACES OF TYPE W

This chapter contains an exposition of the theory of test function spaces of type W, which together with the spaces of type S (Volume 2, Chapter IV) will be used in Chapters II and III of the present volume for the study of Cauchy's problem. The results contained in the present chapter have been summarized without proofs in Appendix 2 to Chapter IV of Volume 2.

The spaces of type W are analogous to spaces of type S, corresponding to values $\alpha < 1$ and $\beta < 1$, but due to the use of arbitrary convex functions in place of powers, these spaces are capable of a more precise description of the peculiarities of growth (or decrease) at infinity.

In the same manner as for spaces of type S, for simplicity we shall first treat the case of one independent variable. The modifications which are necessitated by considering several independent variables are indicated in Section 4.

1. Definitions

1.1. The Spaces W_M

Let $\mu(\xi)$ $(0 \leqslant \xi < \infty)$ denote a continuous increasing function, such that $\mu(0) = 0$, $\mu(\infty) = \infty$. We define for $x \geqslant 0$

$$M(x) = \int_0^x \mu(\xi)\, d\xi. \tag{1}$$

The function $M(x)$ is an increasing convex continuous function, with $M(0) = 0$, $M(\infty) = \infty$. Since $\mu(\xi)$ increases with the increase of ξ, so does its average ordinate $x^{-1}M(x)$, so that for arbitrary positive x_1 and x_2, we have

$$\frac{1}{x_1} M(x_1) \leqslant \frac{1}{x_1 + x_2} M(x_1 + x_2),$$

$$\frac{1}{x_2} M(x_2) \leqslant \frac{1}{x_1 + x_2} M(x_1 + x_2).$$

Multiplying the first inequality by x_1, the second by x_2, and adding, we obtain the fundamental (convexity) inequality

$$M(x_1) + M(x_2) \leqslant M(x_1 + x_2). \tag{2}$$

In particular, for any $x \geqslant 0$

$$2M(x) \leqslant M(2x). \tag{3}$$

Further, we define the function $M(x)$ for negative x by means of the equality

$$M(-x) = M(x).$$

Note that since the derivative $\mu(x)$ of the function $M(x)$ is unbounded for $x \to +\infty$, the function $M(x)$ itself will *grow faster than any linear function* as $|x| \to \infty$.

We shall denote by W_M the set of all infinitely differentiable functions $\varphi(x)$ $(-\infty < x < \infty)$ satisfying the inequalities

$$|\varphi^{(q)}(x)| \leqslant C_q e^{-M(ax)} \tag{4}$$

with constants C_q and a which may depend on the function φ.

Since the function $M(x)$ increases faster than any linear function, the function $e^{-M(ax)}$ will decrease faster than any exponential function (i.e., a function of the form $e^{-a|x|}$); thus the test functions $\varphi(x)$ which belong to the space W_M, as well as all their derivatives, decrease at infinity faster than any exponential function.

It is obvious that W_M is a vector space (with the usual operations). We introduce for this space, the following definition of convergence: a sequence $\{\varphi_\nu(x)\}$ is said to *converge to zero* if the functions $\varphi_\nu(x)$ and all their derivatives converge to zero uniformly on any finite interval of the x-axis (such convergence is called *regular convergence*) and in addition the following inequalities hold:

$$|\varphi_\nu^{(q)}(x)| \leqslant C_q\, e^{-M(ax)}, \tag{5}$$

where the constants C_q and a do not depend on ν.

Let us show that the space W_M can be represented as a union of countably normed spaces.

We denote by $W_{M,a}$ the set of all functions from the space W_M which obey the inequalities

$$|\varphi^{(q)}(x)| \leqslant C_q \exp[-M(\bar{a}x)],$$

where the constant $\bar{a}$ can be selected arbitrarily, but smaller than a. In other words, the space $W_{M,a}$ consists of those functions $\varphi(x)$ which satisfy for any $\delta > 0$ the inequalities

$$|\varphi^{(q)}(x)| \leqslant C_{q\delta} \exp[-M[(a-\delta)x]] \qquad (q = 0, 1, 2,...).$$

We define

$$M_p(x) = \exp\left(M\left[a\left(1-\frac{1}{p}\right)x\right]\right) \qquad (p = 2, 3,...). \tag{6}$$

The functions $M_p(x)$ form an increasing sequence, $M_p(x) \leqslant M_{p+1}(x)$, and the functions $\varphi(x) \in W_{M,a}$ can be characterized as infinitely differentiable functions for which the norm

$$\|\varphi\|_p = \sup_{\substack{x \\ |q| \leqslant p}} M_p(x) \, |\varphi^{(q)}(x)| \tag{7}$$

is finite for arbitrary p. This shows that the space $W_{M,a}$ coincides with the space $K\{M_p\}$ defined in Volume 2, Chapter II, Section 1, with a fixed sequence of weight functions (6). Therefore, all the results referring to the spaces $K\{M_p\}$ may be applied to the space $W_{M,a}$. It is thus a complete countably normed space with the norms (7). We show that it is a perfect space. The condition (p), which is sufficient for the space $K\{M_p\}$ to be perfect (Volume 2, Chapter II, Section 2), consists in the existence for any p of a number $p' > p$, such that

$$\lim_{|x| \to \infty} \frac{M_p(x)}{M_{p'}(x)} = 0.$$

In our case, due to the convexity inequality, we have for any $p' > p$

$$M\left[a\left(1-\frac{1}{p}\right)x\right] + M\left[a\left(\frac{1}{p}-\frac{1}{p'}\right)x\right] \leqslant M\left[a\left(1-\frac{1}{p'}\right)x\right],$$

and consequently

$$\frac{M_p(x)}{M_{p'}(x)} = \exp\left(M\left[\left(1-\frac{1}{p}\right)ax\right] - M\left[\left(1-\frac{1}{p'}\right)ax\right]\right)$$
$$\leqslant \exp\left(-M\left[\left(\frac{1}{p}-\frac{1}{p'}\right)ax\right]\right) \to 0,$$

as required for the proof.

According to the results of Volume 2, Chapter II, Section 2, a sequence $\varphi_\nu(x) \in W_{M,a}$ converges to zero if and only if the sequence $\varphi_\nu(x)$ is regularly convergent (i.e., the functions $\varphi_\nu^{(q)}(x)$, for any q, converge uniformly to zero on any interval $|x| \leqslant x_0 < \infty$) and the norms $\|\varphi_\nu\|_p$ are bounded for any p.

The union of the spaces $W_{M,a}$ with all indices $a = 1, \frac{1}{2}, \ldots$ obviously coincides with the space W_M. The convergence to zero in the space W_M, as described above, is the convergence to zero in one of the spaces $W_{M,a}$, and thus coincides with the concept of convergence defined in W_M considered as a union of countably normed spaces.

We now define bounded sets in the space W_M. According to the general definition of bounded sets in a union of countably normed spaces, set $A \subset W_M$ is said to be *bounded*, if it is entirely contained within one of the $W_{M,a}$ and is bounded in this space. In other words, the set $A \subset W_M$ is bounded if all functions $\varphi(x) \in A$ satisfy the inequalities (4) with the same constants C_q and a. In particular, a sequence $\varphi_\nu(x) \in W_M$ converges to zero if (1) it converges to zero regularly, (2) it is bounded.

Example 1. Let $M(x) = x^{1/\alpha}$ $(x > 0)$, with $\alpha < 1$; then $\mu(\xi) = (1/\alpha)\, \xi^{(1/\alpha)-1}$. The corresponding space W_M consists of infinitely differentiable functions $\varphi(x)$, satisfying the inequalities

$$|\varphi^{(q)}(x)| \leqslant C_q e^{-a|x|^{1/\alpha}}$$

for certain C_q and a which depend on φ. Obviously, this space coincides with the space S_α (Volume 2, Chapter IV, Section 1).

Example 2. Let $\mu(\xi) = \ln(\xi + 1)(\xi \geqslant 0)$; then, for $x \geqslant 0$

$$M(x) = \int_0^x \ln(\xi + 1)\, d\xi = (x + 1)\ln(x + 1) - x.$$

According to the definition, the space W_M consists of the infinitely differentiable functions $\varphi(x)$ which satisfy the inequalities

$$|\varphi^{(q)}(x)| \leqslant C_q \exp(-a[(|x| + 1)\ln(|x| + 1) - |x|]).$$

In this case the functions $\varphi(x)$ admit a simpler description, which can be obtained by means of the following considerations.

Formally one could have constructed a space W_M starting from any nonnegative continuous function $M(x)$ (without taking into account whether this function has the special form (1); later we shall make use of this special form), by means of the definition (4). In this case, one may obtain the same space for different functions $M_1(x)$ and $M_2(x)$, $W_{M_1} \equiv W_{M_2}$. We indicate a simple sufficient condition for this equality to hold. Assume that the functions $M_1(x)$ and $M_2(x)$ satisfy for sufficiently large $x \geqslant 0$ the inequality

$$M_1(\gamma_1 x) \leqslant M_2(\gamma_2 x) \tag{8}$$

with some positive constants γ_1 and γ_2. Then we can assert that the inclusion

$$W_{M_1} \supset W_{M_2}$$

holds. Indeed, Eq. (8) can be replaced by an inequality, valid for all values of $x \geqslant 0$, by adding a suitable constant

$$M_1(\gamma_1 x) \leqslant M_2(\gamma_2 x) + \gamma_3 .$$

Hence, if $\varphi(x) \in W_{M_2}$, we have

$$| \varphi^{(q)}(x) | \leqslant C_q \exp[-M_2(ax)] \leqslant C_q' \exp[M_1(a'x)], \tag{9}$$

with $a' = a(\gamma_1/\gamma_2)$, $C_q' = C_q e^{\gamma_3}$; thus $\varphi \in W_{M_1}$.

Moreover, the inequality (9) shows that if the sequence $\varphi_\nu(x)$ converges to zero in the sense of W_{M_2}, it does so also in the sense of W_{M_1}, since one can choose the constants a' and C_q' in the inequalities (9) for the functions $\varphi_\nu(x)$ together with the constants a and C_q independently of ν.

Further, if the functions $M_1(x)$ and $M_2(x)$ are such that for sufficiently large $x \geqslant 0$

$$M_1(\gamma_1 x) \leqslant M_2(\gamma_2 x) \leqslant M_1(\gamma_1' x), \tag{10}$$

then both inclusions $W_{M_1} \supset W_{M_2}$ and $W_{M_2} \supset W_{M_1}$ hold, and thus $W_{M_1} \equiv W_{M_2}$, as sets; it is also obvious that the convergence in W_{M_1} coincides with the convergence in W_{M_2}. Two functions $M_1(x)$ and $M_2(x)$ satisfying the inequality (10) will be called *equivalent*; we have seen that equivalent functions define the same space.

The function $(x + 1) \ln(x + 1) - x$, which appears in Example 2, is equivalent to the function $x \ln x$ (which does not satisfy the definition (1)); consequently, the corresponding test function space is also defined by means of the inequalities

$$| \varphi^{(q)}(x) | \leqslant C_q \exp(-a \mid x \mid \ln \mid x \mid)$$

and the corresponding definition of convergence.

1.2. The Spaces W^{Ω}

Let $\omega(\eta)$ $(0 \leqslant \eta < \infty)$ denote an increasing continuous function with $\omega(0) = 0$, $\omega(\infty) = \infty$; for $y \geqslant 0$ we define

$$\Omega(y) = \int_0^y \omega(\eta)\, d\eta. \tag{1}$$

The properties of the function $\Omega(y)$ are entirely analogous to those of the function $M(x)$, introduced in Section 1.1; in particular, the convexity inequalities hold:

$$\Omega(y_1) + \Omega(y_2) \leqslant \Omega(y_1 + y_2), \tag{2}$$

$$2\Omega(y) \leqslant \Omega(2y) \tag{3}$$

We further define

$$\Omega(-y) = \Omega(y).$$

We shall denote by W^{Ω} the set of all entire analytic functions $\varphi(z)$ $(z = x + iy)$, which satisfy the inequalities

$$| z^k\varphi(z) | \leqslant C_k e^{\Omega(by)} \tag{4}$$

where the constants C_k and b depend on the function φ.

It is obvious that W^{Ω} is a vector space with the usual definitions of the operations (over the field of complex numbers). We introduce for this vector space the following definition of convergence: a sequence $\varphi_\nu(z) \in W^{\Omega}$ is said to *converge to zero* if the functions $\varphi_\nu(z)$ converge uniformly to zero in any bounded domain of the z-plane (this will be called regular convergence) and in addition satisfy the inequalities

$$| z^k\varphi_\nu(z) | \leqslant C_k e^{\Omega(by)},$$

where the constants C_k and b do not depend on the index ν.

The space W^{Ω} can be represented as a union of countably normed spaces. Indeed, let us denote by $W^{\Omega,b}$ the set of those functions in W^{Ω} which satisfy the inequalities

$$| z_k\varphi(z) | \leqslant C_k \exp[\Omega(\bar{b}y)],$$

where $\bar{b}$ can be any constant larger than b. In other words, the set $W^{\Omega,b}$ consists of those entire functions which for any $\rho > 0$ satisfy the inequalities

$$|z_k\varphi(z)| \leqslant C_{k\rho} \exp[\Omega[(b + \rho)y]).$$