Undergraduate Texts in Mathematics

James G. Simmonds

A Brief on Tensor Analysis

Second Edition

张量分析简论 第2版

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(continued after index)

To the memory of my father,
My first and greatest teacher

Preface to the Second Edition

There are three changes in the second edition. First, with the help of readers and colleagues—thanks to all—I have corrected typographical errors and made minor changes in substance and style. Second, I have added a few more Exercises, especially at the end of Chapter 4. Third, I have appended a section on Differential Geometry, *the* essential mathematical tool in the study of two-dimensional structural shells and four-dimensional general relativity.

JAMES G. SIMMONDS

Preface to the First Edition

When I was an undergraduate, working as a co-op student at North American Aviation, I tried to learn something about tensors. In the Aeronautical Engineering Department at MIT, I had just finished an introductory course in classical mechanics that so impressed me that to this day I cannot watch a plane in flight—especially in a turn—without imaging it bristling with vectors. Near the end of the course the professor showed that, if an airplane is treated as a rigid body, there arises a mysterious collection of rather simple-looking integrals called the components of the moment of inertia tensor. Tensor—what power those two syllables seemed to resonate. I had heard the word once before, in an aside by a graduate instructor to the *cognoscenti* in the front row of a course in strength of materials. "What the book calls stress is actually a tensor. . . ."

With my interest twice piqued and with time off from fighting the brushfires of a demanding curriculum, I was ready for my first serious effort at self-instruction. In Los Angeles, after several tries, I found a store with a book on tensor analysis. In my mind I had rehearsed the scene in which a graduate student or professor, spying me there, would shout, "You're an undergraduate. What are you doing looking at a book on tensors?" But luck was mine: the book had a plain brown dust jacket. Alone in my room, I turned immediately to the definition of a tensor: "A 2nd order tensor is a collection of n^2 objects that transform according to the rule . . ." and thence followed an inscrutable collection of superscripts, subscripts, overbars, and partial derivatives. A pedagogical disaster! Where was the connection with those beautiful, simple, boldfaced symbols, those arrows that I could visualize so well?

I was not to find out until after graduate school. But it is my hope that, with this book, you, as an undergraduate, may sail beyond that bar on which I once foundered. You will find that I take nearly three chapters to prepare you for the shock of the tensor transformation formulas. I don't try to hide them—they're the only equations in the book that are boxed. But long before, about halfway through Chapter 1, I tell you what a 2nd order tensor *really* is—a linear operator that sends vectors into vectors. If you apply the stress tensor to the unit normal to a plane through a point in a body, then out comes the stress vector, the force/area acting across the plane at that point. (That the stress vector is linear in the unit normal, i.e., that a stress tensor even exists, is a gift of nature; nonlinearity is more often the rule.) The subsequent "débauche des indices" that follows this tidy definition of a 2nd order tensor is the result of exposing the gears of a machine for grinding out the workings of a tensor. Abolish the machine and there is no hope of producing numerical results except in the simplest of cases.

This book falls into halves: Algebra and Calculus. The first half of the first half (Chapter 1) emphasizes concepts. Here, I have made a special effort to relate the mathematical and physical notions of a vector. I acknowledge my debt to Hoffman's intriguing little book, *About Vectors* (Dover, 1975). (But there are points where we differ—I disagree with his contention that vectors cannot represent finite rotations.) Chapter 2 deals mostly with the index apparatus necessary to represent and manipulate vectors and tensors in general bases. Chapter 3, through the vehicle of Newton's law of motion, introduces moving frames and the Christoffel symbols. To help keep the basic kinematic ideas and their tensor generalizations in mind simultaneously, I list a number of equations in dual form, a device that I have found successful in the classroom. The last chapter starts with a homely example of the gradient and builds to the covariant derivative. Throughout this chapter there are applications to continuum mechanics. Although the basic equations (excluding electricity and magnetism) were known by the 1850's, it was only under the spur of general relativity that tensor analysis began to diffuse into this older field. (In my own specialty, shell theory, tensor analysis did not appear until the early 1940's, in the Soviet literature, even though the underlying theory of surfaces and their tensor description had been central to the understanding of general relativity.)

I have provided no systematic lists of grad, div, curl, etc. in various coordinate systems. Such useful information can be found in Magnus, Oberhettinger, and Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd enlarged edition, Chapter XII, Springer-Verlag 1966; or in Gradshteyn and Ryzhik, *Tables of Integrals, Series and Products*, 4th edition, corrected and enlarged, Academic Press, 1980.

It is a happy thought that much of the drudgery involved in expanding equations and verifying solutions in specific coordinate systems can now be done by computers, programmed to do symbol manipulation. The interested reader should consult "Computer Symbolic Math in Physics Education," by D. R. Stoutemyer, *Am. J. Phys.*, vol. 49 (1981), pp. 85–88, or "A Review of Algebraic Computing in General Relativity," by R. A. d'Inverno, Chapter 16 of *General Relativity and Gravitation*, vol. 1, ed. A. Held, Plenum Press, N.Y. and London, 1980.

I am pleased to acknowledge the help of three friends: Mark Duva, a former student, who, in his gracious but profound way, let me get away with nothing in class; Bruce Chartres, who let me filter much of this book through his fine mind; and Ernst Soudek, who, though not a native speaker, tuned the final manuscript with his keen ear for English.

Finally, my thanks to Carolyn Duprey and Ruth Nissley, who typed the original manuscript, and then with patience and good humor, retyped what must have seemed to be hundreds of petty changes.

JAMES G. SIMMONDS

Contents

CHAPTER I

Introduction: Vectors and Tensors

The magic of this theory will hardly fail to impose itself on anybody who has truly understood it; it represents a genuine triumph of the method of absolute differential calculus, founded by Gauss, Riemann, Christoffel, Ricci and Levi–Civita.[1]

This little book is about tensor analysis, as Einstein's philosophers' stone, the absolute differential calculus, is called nowadays. I have written it, though, with an eye not toward general relativity, but to *continuum mechanics*, a more modest theory that attempts to predict the gross behavior of "the masses of matter we see and use from day to day: air, water, earth, flesh, wood, stone, steel, concrete, glass, rubber, ..."[2]

Continuum mechanics is a limiting case of general relativity; yet it is best treated on its own merits. Viewed thus, there is a fundamental difference at the foundations of the two theories. The geometry of continuum mechanics is that of *three-dimensional Euclidean space* (E_3 for short) and *the real line*, R. The geometry of general relativity is that of a *four-dimensional Riemannian manifold.* (A sphere is a two-dimensional Riemannian manifold.) To those who will settle for nothing less than a complete understanding of general relativity (and who, therefore, will want to consult *Gravitation*, by Misner, Thorne, and Wheeler), take heart. From the tools that we shall fashion comes the gear to scale that pinnacle. And to those content to cultivate the garden of continuum mechanics, let me say that, embedded within it, are intrinsically

[1] Albert Einstein, "Contribution to the Theory of General Relativity", 1915; as quoted and translated by C. Lanczos in *The Einstein Decade*, p. 213.

[2] Truesdell and Noll, *The Non-Linear Field Theories of Mechanics*, p. 1. Two outstanding introductory texts on continuum mechanics are *A First Course in Rational Continuum Mechanics*, 2nd ed, by Truesdell and *Continuum Mechanics* by Chadwick.

curved two-dimensional continua, called shells, that in dwarf form exhibit nearly all of the mathematical foliage found in full-flowered general relativity.

In attempting to give mathematical form to the laws of mechanics, we face a dichotomy. On the one hand, if physical events and entities are to be quantified, then a (reference) *frame* and a *coordinate system* within that frame must be introduced.[3] On the other hand, as a frame and coordinates are mere scaffolding, it should be possible to express the laws of physics in frame- and coordinate-free form, i.e. in *invariant* form. Indeed this is the great program of general relativity.

In continuum mechanics, however, there are exceptional frames called *inertial*; Newton's Law of motion for a particle—force equals mass times acceleration—holds only in such frames.[4] A basic concern of continuum mechanics is therefore how laws such as Newton's change from one frame to another.[5] Save for Exercise 4.24, we shall not analyze changes of frame. Rather we shall study how, within a *fixed* frame, the mathematical *representation* of a physical object or law changes when one coordinate system (say Cartesian) is replaced by another (say spherical).

In what follows, I have assumed that you remember some of the plane and solid geometry that you once learned and that you have seen a bit of vector algebra and calculus. For conciseness, I have omitted a number of details and examples that you can find in texts devoted to vectors. At the same time I have emphasized several points, especially those concerning the physical meaning of vector addition and component representation, that are *not* found in most conventional texts. The exercises at the end of each chapter are intended to amplify and to supplement material in the text.

[3] A frame is a mathematical representation of a physical apparatus which assigns to each event e in the physical world $\mathscr{W}$ a unique *place* (i.e. point) in E_3 and a unique *instant* on the real line R. I like to imagine an idealized, all-seeing stereographic video camera mounted on 3 rigid, mutually perpendicular rods. The rods have knife edges that intersect at a point and one of the rods carries a scratch to fix a unit of length. The 3 knife edges (indefinitely prolonged) are represented by a right-handed Cartesian reference frame $0xyz$ in E_3, and one instant (arbitrarily chosen) is taken as the origin of the real line. The exposed tape is a physical realization of a *framing* (to use the terminology of Truesdell, *op. cit.*), i.e. a map f from $\mathscr{W}$ to $E_3 \times R$.

A coordinate system in a frame assigns to each place a unique triple of real numbers (u, v, w) called spatial coordinates and to each instant a unique number t call the time.

[4] Inertial frames are also special, but in a different way, in general relativity where frames *are* coordinate systems! (and physics *is* geometry). An inertial frame may be introduced in general relativity in the same way as a two-dimensional Cartesian coordinate system may be introduced in an arbitrarily small neighborhood of a point on a sphere.

[5] To change frames means, for example, to tape the world with a *copy* of our super camera. If the cameras are in relative motion, then the two exposed tapes f and f_* will map the same event e into different places P and P_* *in* E_3 and into different instants T and T_* on R. Of course, the two sets of knife edges are represented by the same frame $0xyz$ and the cameras run at the same rate. This *change of frame* is a special type of time-dependent map of $E_3 \times R$ into itself that preserves the distance and elapsed time between two events. When the elapsed time is zero, this transformation has the same form as a *rigid body motion*. See Exercise *4.19* and *Truesdell, op. cit.*

Three-Dimensional Euclidean Space

Three-dimensional Euclidean space, E_3, may be characterized by a set of axioms that expresses relationships among primitive, undefined quantities called points, lines, etc.[6] These relationships so closely correspond to the results of ordinary measurements of distance in the physical world that, until the appearance of general relativity, it was thought that Euclidean geometry was *the* kinematic model of the universe.

Directed Line Segments

Directed line segments, or *arrows*, are of fundamental importance in Euclidean geometry. Logically, an arrow is an ordered pair of points, (A, B). A is called the *tail* of the arrow and B the *head*. It is customary to represent such an arrow typographically as $\overline{AB}$, and pictorially as a line segment from A to B with an arrow head at B. (To avoid crowding, the arrow head may be moved towards the center of the segment). Assigning a length to an arrow or multiplying it by a real number (holding the tail fixed) are precisely defined operations in E_3.

Two arrows are said to be *equivalent* if one can be brought into coincidence with the other by a parallel translation.[7] In Fig. 1.1, $\overline{AB}$ and $\overline{CD}$ are equivalent, but neither $\overline{AB}$ and $\overline{EF}$ nor $\overline{AB}$ and $\overline{GH}$ are.

The set of *all* arrows equivalent to a given arrow is called the (geometric) *vector* of that arrow and is usually denoted by a symbol such as $\mathbf{v}$. A vector is an example of an *equivalence class* and, by convention, a vector is represented by any one of its arrows.

Equivalence classes are more familiar (and more useful) than you may realize. Suppose that we wish to carry out, on a computer, exact arithmetic

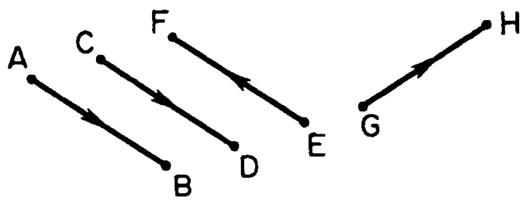

Figure 1.1.

[6] This was Hilbert's program: reduce geometry to a branch of logic. No pictures allowed! See, for example, the discussion at the end of Eisenhart's *Analytic Geometry*. Our approach, however, shall be informal and visual.

[7] A definition that makes no sense on a sphere. Why?

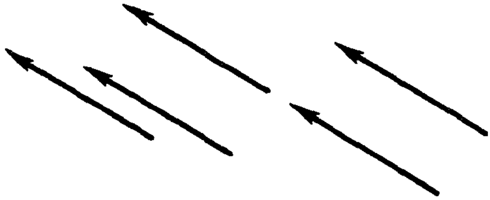

Figure 1.2.

operations on rational numbers. Then, for example, $\frac{2}{3}$ must be read in as the ordered pair of integers (2, 3). We test for the equivalence of two ordered pairs of integers (a, b) and (c, d) stored within the computer by checking to see if $ad = bc$. In doing so, we are tacitly using the definition of a rational number a/b as the equivalence class of all ordered pairs of integers (c, d) such that $ad = bc$.

In practice, it is expedient (and rarely causes problems) to confound a "number", such as two-thirds, with its various representations e.g., 2/3, 4/6, etc. Likewise, we shall be using the term "vector" when we mean one of its arrows (and vice versa), relying on context for the proper interpretation. Thus in Fig. 1.2 we call any one of the equivalent arrows "the vector $\mathbf{v}$."

The length of a vector $\mathbf{v}$ is denoted by $|\mathbf{v}|$ and defined to be the length of any one of its arrows. The zero vector, $\mathbf{0}$, is the unique vector having zero length. We call the unit vector

$$\bar{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|, \qquad \mathbf{v} \neq \mathbf{0}, \tag{1.1}$$

the direction of $\mathbf{v}$; $\mathbf{0}$ has no direction.

We may choose, *arbitrarily*, a point 0 in E_3 and call it the *origin*. The vector $\mathbf{x}$ (of the arrow) from 0 to a point P is called the *position* of P. We shall sometimes write $P(\mathbf{x})$ as shorthand for "the point with position $\mathbf{x}$."

Addition of Two Vectors

Addition of two vectors $\mathbf{u}$ and $\mathbf{v}$ may be defined in two equivalent ways.[8]

A. The Head-to-Tail-Rule (Fig. 1.3a). Take any arrow representing $\mathbf{u}$, say $\overline{AB}$. For this choice there is a unique arrow $\overline{BC}$ representing $\mathbf{v}$; $\mathbf{u} + \mathbf{v}$ is defined to be the vector of the arrow $\overline{AC}$. This definition is convenient if one wishes to add a string of vectors (Exercise 1.1), but commutativity is not obvious. For reasons of symmetry it may be preferable to use the following.

[8] The equivalence and uniqueness of the two definitions can be proved from the postulates of Euclidean geometry.