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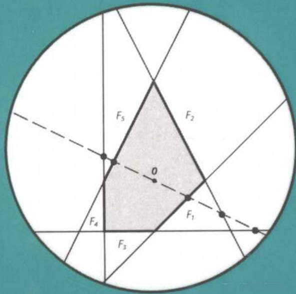
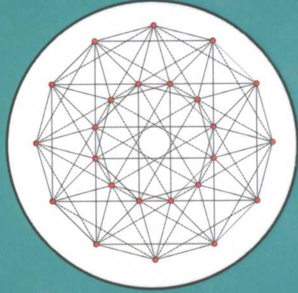
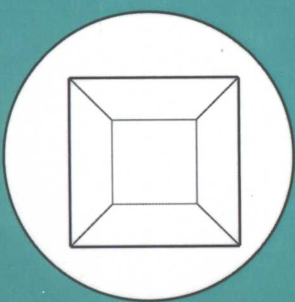
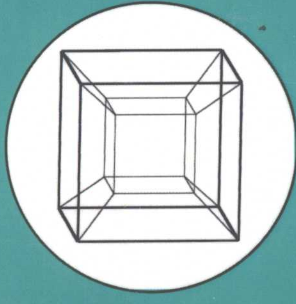


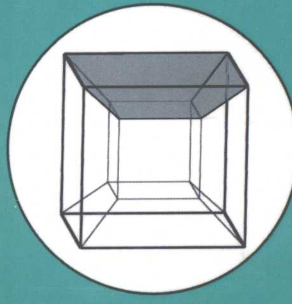


INTRODUCTION TO TOPOLOGY AND GEOMETRY

Second Edition







SAUL STAHL
CATHERINE STENSON

WILEY

INTRODUCTION TO TOPOLOGY AND GEOMETRY

Second Edition

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To Denise, with love from Saul.

To my family, with love from Cathy.

PREFACE

This book is intended to serve as a text for a two-semester undergraduate course in topology and modern geometry. It is devoted almost entirely to the geometry of the last two centuries. In fact, some of the subject matter was discovered only within the last two decades. Much of the material presented here has traditionally been part of the realm of graduate mathematics, and its presentation in undergraduate courses necessitates the adoption of certain informalities that would be unacceptable at the more advanced levels. Still, all of these informalities either were used by the mathematicians who created these disciplines or else would have been accepted by them without any qualms.

The first four chapters aim to serve as an introduction to topology. Chapter 1 provides an informal explanation of the notion of homeomorphism. This naive introduction is in fact sufficient for all the subsequent chapters. However, the instructor who prefers a more rigorous treatment of basic topological concepts such as homeomorphisms, topologies, and metric spaces will find it in Chapter 10.

The second chapter emphasizes the topological aspects of graph theory, but is not limited to them. This material was selected for inclusion because the accessible nature of some of its results makes it the pedagogically perfect vehicle for the transition from the metric Euclidean geometry the students encountered in high school to the combinatorial thinking that underlies the topological results of the subsequent chapters. The focal issue here is planarity: Euler's Theorem, coloring theorems, and the Kuratowski Theorem.

Chapter 3 presents the standard classifications of surfaces of both the closed and bordered varieties. The Euler–Poincaré equation is also proved.

Chapter 4 is concerned with the interplay between graphs and surfaces—in other words, graph embeddings. In particular, a procedure is given for settling the question of whether a given graph can be embedded on a given surface. Polygonal (2-cell) embeddings and their rotation systems are discussed. The notion of covering surfaces is introduced via the construction of voltage graphs.

The theory of knots and links has recently received tremendous boosts from the work of John Conway, Vaughan Jones, and others. Much of this work is easily accessible, and some has been included in Chapter 5: the Conway–Gordon–Sachs Theorem regarding the intrinsic linkedness of the graph K_6 in $\mathbb{R}^3$ and the invariance of the Jones polynomial. While this discipline is not, properly speaking, topological, connections to the topology of surfaces are not lacking. Knot theory is used to prove the nonembeddability of nonorientable surfaces in $\mathbb{R}^3$, and surface theory is used to prove the nondecomposability of trivial knots. The more traditional topic of labelings is also presented.

The next three chapters deal with various aspects of differential geometry. The exposition is as elementary as the author could make it and still meet his goals: explanations of Gauss's Total Curvature Theorem and hyperbolic geometry. The geometry of surfaces in $\mathbb{R}^3$ is presented in Chapter 6. The development follows that of Gauss's *General Investigations of Curved Surfaces*. The subtopics include Gaussian curvature, geodesics, sectional curvatures, the first fundamental form, intrinsic geometry, and the Total Curvature Theorem, which is Gauss's version of the famed Gauss–Bonnet formula. Some of the technical lemmas are not proved but are instead supported by informal arguments that come from Gauss's monograph. A considerable amount of attention is given to polyhedral surfaces for the pedagogical purpose of motivating the key theorems of differential geometry.

The elements of Riemannian geometry are presented in Chapter 7: Riemann metrics, geodesics, isometries, and curvature. The numerous examples are also meant to serve as a lead-in to the next chapter.

The eighth chapter deals with hyperbolic geometry. Neutral geometry is defined in terms of Euclid's axiomatization of geometry and is described in terms of Euclid's first 28 propositions. Various equivalent forms of the parallel postulates are proven, as well as the standard results regarding the sum of the angles of a neutral triangle. Hyperbolic geometry is also defined axiomatically. Poincaré's half-plane geometry is developed in some detail as an instance of the Riemann geometries of the previous chapter and is demonstrated to be hyperbolic. The isometries of the half-plane are described both algebraically and geometrically.

The ninth chapter is meant to serve as an introduction to algebraic topology. The requisite group theory is summarized in Appendix B. The focus is on the derivation of fundamental groups, and the development is based on Poincaré's own exposition and makes use of several of his examples. The reader is taught to derive presentations for the fundamental groups of the punctured plane, closed surfaces, 3-manifolds, and knot complements. The chapter concludes with a discussion of the Poincaré Conjecture.

The tenth chapter serves a dual purpose. On the one hand it aims to acquaint the reader with the elegant topic of general topology and the joys of sequence chasing. On the other hand, it contains the rigorous definitions of a variety of fundamental concepts that were only informally defined in the previous chapters. In terms of mathematical maturity, this is probably the most demanding part of the book.

The last chapter is devoted to the study of polytopes. Following an introduction, attention is given to the graphs of polytopes, regular polytopes, and the enumeration of faces of polytopes.

Wherever appropriate, historical notes have been interspersed with the exposition. Care was taken to supply many exercises that range from the routine to the challenging. Middle-level exercises were hard to come by, and the author welcomes all suggestions.

An Instructor's solution manual is available upon request from Wiley.

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S. S.

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C. S.

CONTENTS

CHAPTER 1

INFORMAL TOPOLOGY

In this chapter the notion of a topological space is introduced, and informal ad hoc methods for identifying equivalent topological spaces and distinguishing between nonequivalent ones are provided.

The last book of Euclid's opus *Elements* is devoted to the construction of the five Platonic solids pictured in Figure 1.1. A fact that Euclid did not mention is that the counts of the vertices, edges, and faces of these solids satisfy a simple and elegant relation. If these counts are denote by v, e, and f, respectively, then

$$v - e + f = 2. \tag{1}$$

Specifically, for these solids we have:

Cube:	$8 - 12 + 6 = 2.$
Octahedron:	$6 - 12 + 8 = 2.$
Tetrahedron:	$4 - 6 + 4 = 2.$
Dodecahedron:	$20 - 30 + 12 = 2.$
Icosahedron:	$12 - 30 + 20 = 2.$

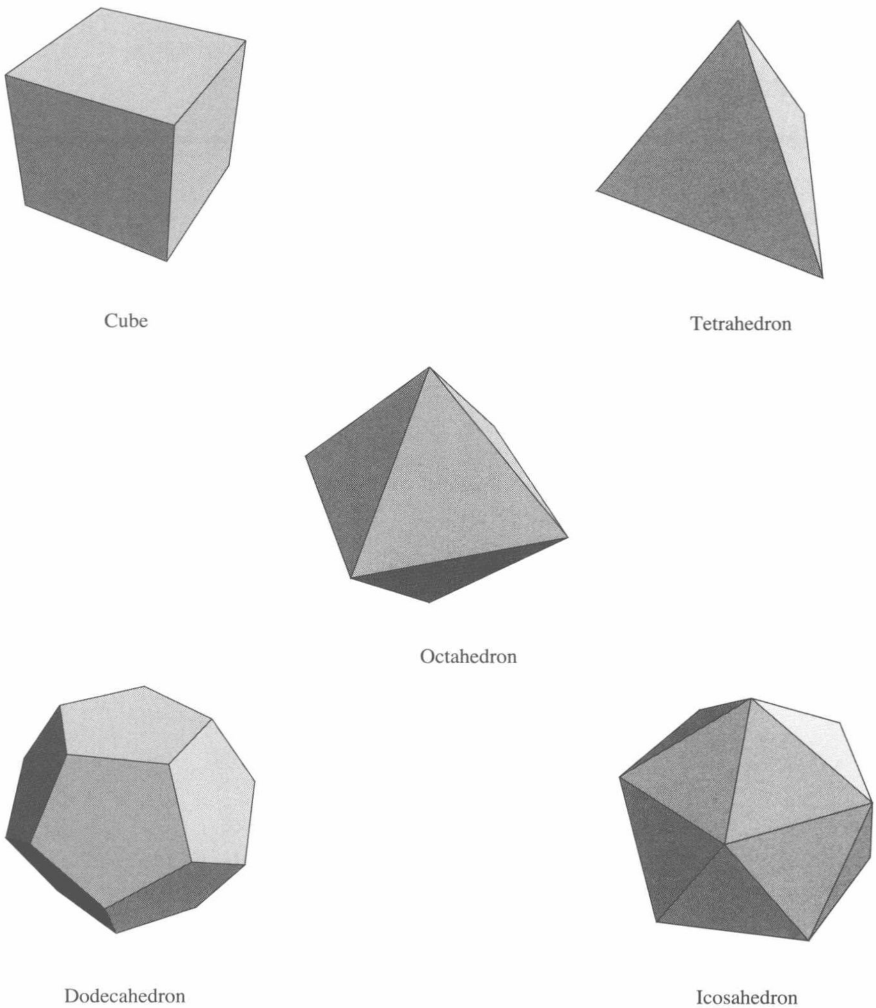

Figure 1.1 The Platonic solids.

A Platonic solid is defined by the specifications that each of its faces is the same regular polygon and that the same number of faces meet at each vertex. An interesting feature of Equation (1) is that while the Platonic solids depend on the notions of length and straightness for their definition, these two aspects are absent from the equation itself. For example, if each of the edges of the cube is either shrunk or extended by some factor, whose value may vary from edge to edge, a lopsided cube is obtained (Fig. 1.2) for which the equation still holds by virtue of the fact that it holds for the (perfect) cube. This is also clearly true for any similar modification of the other four Platonic solids. The fact of the matter is that Equation (1) holds

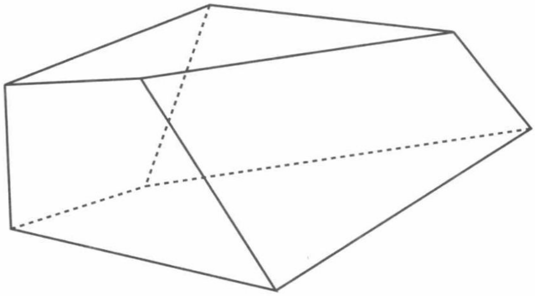

Figure 1.2 A lopsided cube.

not only for distorted Platonic solids, but for all solids as well, provided these solids are carefully defined. Thus, for the three solids of Figure 1.3 we have respectively $5-8+5=2$, $6-9+5=2$, and $7-12+7=2$. The applicability of Equation (1) to all such solids was first noted by Leonhard Euler (1707–1783) in 1758, although some historians contend that this equation was presaged by certain observations of René Descartes (1596–1650).

Euler's equation remains valid even after the solids are subjected to a wider class of distortions which result in the curving of their edges and faces (see Figure 1.4). One need simply relax the definition of edges and faces so as to allow for any non-self intersecting curves and surfaces. Soccer balls and volleyballs, together with the patterns formed by their seams, are examples of such curved solids to which Euler's equation applies. Moreover, it is clear that the equation still holds after the balls are deflated.

Topology is the study of those properties of geometrical figures that remain valid even after the figures are subjected to distortions. This is commonly expressed by saying that topology is *rubber-sheet geometry*. Accordingly, our necessarily informal definition of a *topological space* identifies it as any subset of space from which the notions of straightness and length have been abstracted; only the aspect of contiguity remains. Points, arcs, loops, triangles, solids (both straight and curved), and the surfaces of the latter are all examples of topological spaces. They are, of course, also geometrical objects, but topology is only concerned with those aspects of their ge-

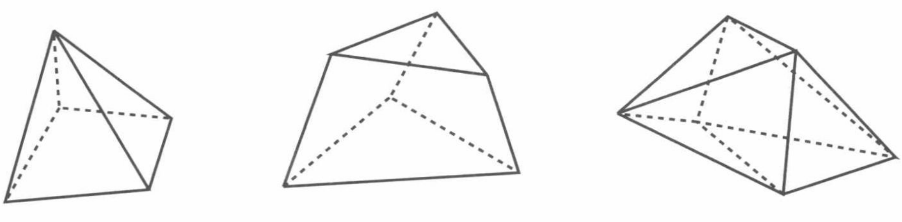

Figure 1.3 Three solids.

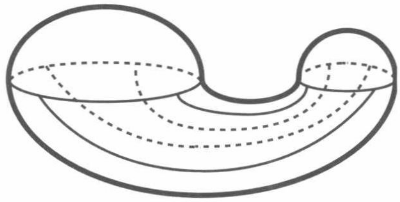

Figure 1.4 A curved cube.

ometry that remain valid despite any translations, elongations, inflations, distortions, or twists.

Another topological problem investigated by Euler, somewhat earlier, in 1736, is known as the *bridges of Koenigsberg*. At that time this Prussian city straddled the two banks of a river and also included two islands, all of which were connected by seven bridges in the pattern indicated in Figure 1.5. On Sunday afternoons the citizens of Koenigsberg entertained themselves by strolling around all of the city's parts, and eventually the question arose as to whether an excursion could be planned which would cross each of the seven bridges exactly once. This is clearly a geometrical problem in that its terms are defined visually, and yet the exact distances traversed in such excursions are immaterial (so long as they are not excessive, of course). Nor are the precise contours of the banks and the islands of any consequence. Hence, this is a topological problem. Theorem 2.2.2 will provide us with a tool for easily resolving this and similar questions.

The notorious Four-Color Problem, which asks whether it is possible to color the countries of every geographical map with four colors so that adjacent countries sharing a border of nonzero length receive distinct colors, is also of a topological nature. Maps are clearly visual objects, and yet the specific shapes and sizes of the countries in such a map are completely irrelevant. Only the adjacency patterns matter.

Every mathematical discipline deals with objects or structures, and most will provide a criterion for determining when two of these are identical, or equivalent. The equality of real numbers can be recognized from their decimal expansions, and two vectors are equal when they have the same direction and magnitude. Topological equivalence is called *homeomorphism*. The surface of a sphere is homeomorphic to

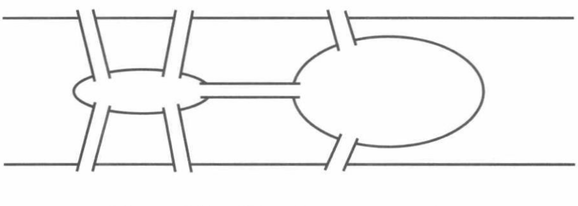

Figure 1.5 The city of Koenigsberg.

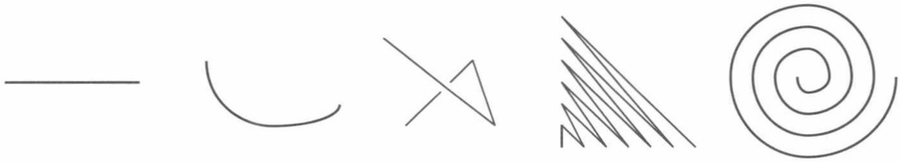

Figure 1.6 Homeomorphic open arcs.

those of a cube, a hockey puck, a plate, a bowl, and a drinking glass. The reason for this is that each of these objects can be deformed into any of the others. Similarly, the surface of a doughnut is homeomorphic to those of an inner tube, a tire, and a coffee mug. On the other hand, the surfaces of the sphere and the doughnut are not homeomorphic. Our intuition rejects the possibility of deforming the sphere into a doughnut shape without either tearing a hole in it or else stretching it out and juxtaposing and pasting its two ends together. Tearing, however, destroys some contiguities, whereas juxtaposition introduces new contiguities where there were none before, and so neither of these transformations is topologically admissible. This intuition of the topological difference between the sphere and the doughnut will be confrmed by a more formal argument in Chapter 3.

The easiest way to establish the homeomorphism of two spaces is to describe a deformation of one onto the other that involves no tearing or juxtapositions. Such a deformation is called an *isotopy*. Whenever isotopies are used in the sequel, their existence will be clear and will require no formal justification. Such is the case, for instance, for the isotopies that establish the homeomorphisms of all the open arcs in Figure 1.6, all the loops in Figure 1.7, and all the ankh-like configurations of Figure 1.8. Note that whereas the page on which all these curves are drawn is two-dimensional, the context is definitely three-dimensional. In other words, all our curves (and surfaces) reside in Euclidean 3-space $\mathbb{R}^3$, and the isotopies may make use of all three dimensions.

The concept of isotopy is insufficient to describe all homeomorphisms. There are spaces which are homeomorphic but not isotopic. Such is the case for the two loops in Figure 1.9. It is clear that loop b is isotopic to all the loops of Figure 1.7 above, and it is plausible that loop a is not, a claim that will be justified in Chapter 5. Hence,

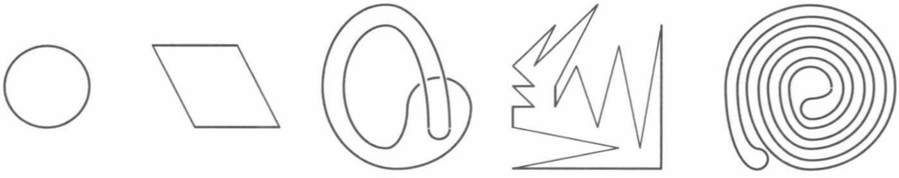

Figure 1.7 Homeomorphic loops.