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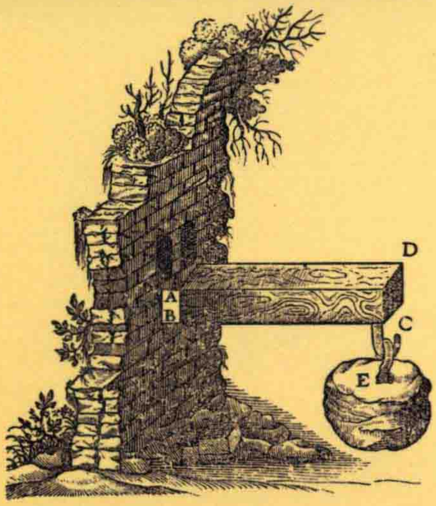
A HISTORY OF THE THEORY OF ELASTICITY AND OF THE STRENGTH OF MATERIALS

VOLUME 2: PART 2
SAINT-VENANT TO LORD KELVIN (2)

ISAAC TODHUNTER
EDITED BY KARL PEARSON



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Physical Sciences

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A History of the Theory of Elasticity and of the Strength of Materials

A distinguished mathematician and notable university teacher, Isaac Todhunter (1820–84) became known for the successful textbooks he produced as well as for a work ethic that was extraordinary, even by Victorian standards. A scholar who read all the major European languages, Todhunter was an open-minded man who admired George Boole and helped introduce the moral science examination at Cambridge. His many gifts enabled him to produce the histories of mathematical subjects which form his lasting memorial. First published between 1886 and 1893, the present work was the last of these. Edited and completed after Todhunter's death by Karl Pearson (1857–1936), another extraordinary man who pioneered modern statistics, these volumes trace the mathematical understanding of elasticity from the seventeenth to the late nineteenth century. Volume 2 (1893) was split into two parts. Part 2 covers the work of Neumann, Kirchhoff, Clebsch, Boussinesq, and Lord Kelvin.

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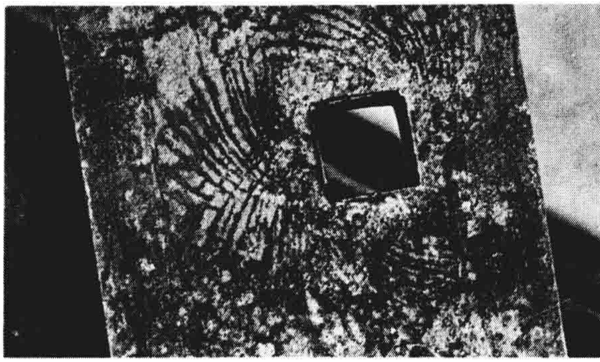

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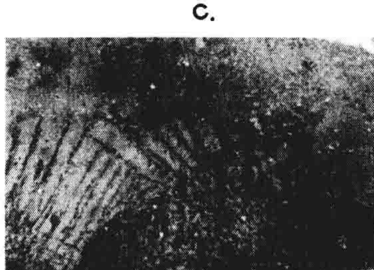

LÜDERS CURVES IN STEEL.

A.—Square punch hole, Weardale Steel.
B.—Bent and punched butt strip of Dredger Bucket, Shelton Steel.
C.—Portion sheared off end of bar of Bush Steel ($2\frac{1}{2}'' \times \frac{3}{8}''$).
Specimens due to Mr. J. B. HUNTER: *see Art.* 1190.

A HISTORY OF

THE THEORY OF ELASTICITY

AND OF

THE STRENGTH OF MATERIALS

FROM GALILEI TO THE PRESENT TIME.

BY THE LATE

ISAAC TODHUNTER, D.Sc., F.R.S.

EDITED AND COMPLETED

FOR THE SYNDICS OF THE UNIVERSITY PRESS

BY

KARL PEARSON, M.A.

PROFESSOR OF APPLIED MATHEMATICS, UNIVERSITY COLLEGE, LONDON,

FORMERLY FELLOW OF KING'S COLLEGE, CAMBRIDGE.

VOL. II. SAINT-VENANT TO LORD KELVIN.

PART II.

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1893.

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ERRATA.

PART II.

Frontispiece, Specimen *B*, *for* butt strip, *read* cutting hip.
p. 282, *l.* 20 *for* Arts. 207—11 *read* Arts. 207*—11*.
p. 286, *l.* 4 from bottom *for* Davier *read* Darier.
p. 341, *l.* 4 from bottom *for* Art. 1863 *read* Art. 1563.

CHAPTER XII.

THE OLDER GERMAN ELASTICIANS: F. NEUMANN, KIRCHHOFF AND CLEBSCH.

SECTION I.

Franz Neumann.

[1192.] WE have already had occasion to deal with three important memoirs of F. Neumann's, which fall into the period occupied by our first volume, and we have now to turn to a work of his which, if only published in 1885, still in substance mainly belongs to the years 1857–8. To Franz Neumann's teaching in Königsberg is due much of the impulse which mathematical physics received in the fifties in Germany; the most distinguished German physicists of the past forty years have been nearly all pupils of Neumann's, and this remark is specially true in the field of elasticity. Of those who attended his lectures on this subject and received probably from him their first stimulus to original investigations, we may name Kirchhoff, Strehlke, Clebsch, Borchardt, Carl Neumann and Voigt as among the more important[1]. Franz Neumann's lectures on elasticity were given in Königsberg at different times from 1857 to 1874, and in 1885 were published under the supervision of O. E. Meyer of Breslau with the title: *Vorlesungen über die Theorie der Elasticität der festen Körper*

[1] O. E. Meyer includes in the list Von der Mühll, Minnigerode, Zöppritz, Gehring, Saalschütz, Wangerin and Baumgarten: see preface to the *Vorlesungen*, S. viii.

und des Lichtäthers. The volume contains xiii + 374 pages, and is based on the notebooks of the brothers L. and O. E. Meyer for the years 1857–60, and those of Baumgarten and W. Voigt for the years 1869–74. According to the Editor the work contains all that was of importance in Neumann's lectures. The exact amount of originality in the several investigations I shall endeavour to point out in the course of my analysis, and I content myself here with the following remarks from the preface:

Zu den Gebieten, mit welchen Professor Neumann sich in jüngeren und späteren Jahren mit besonderer Vorliebe beschäftigt hat, gehört auch die Theorie der Elasticität; es konnte daher nicht fehlen, dass seine Vorlesungen über diesen Gegenstand häufig eigene Arbeiten betrafen. Seinem ausgesprochenen Wunsche, dass alle in verschiedenen Semestern vorgetragenen eigenen Untersuchungen in dieses Werk aufgenommen werden sollten, bin ich gern soweit nachgekommen, als es mir zu erreichen möglich war (S. v–vi).

The work is divided into twenty-one sections of which we note the important points in the following articles.

[1193.] In Section 1, *Einleitung* (S. 1–7), we have first some remarks on the origin of the theory of elasticity. Neumann attributes it not so much to a development from the isolated problems of Bernoulli and Euler as to the impulse given by Fresnel's new theory of light. He says:

Die exacte Beurtheilung seiner Beobachtungen führte Fresnel zu Thatsachen, welche im geraden Widerspruch standen zu den anerkannten Principien der Wellenbewegung in elastischen Medien. In der Schallwelle ist die Bewegung der Theilchen parallel dem Strahl, die Welle eine longitudinale; Fresnel fand, dass in der Lichtwelle jene Bewegung senkrecht gegen den Strahl gerichtet, die Welle also eine transversale ist, und doch soll der Unterschied der Eigenschaften beider Medien, der Luft und des Lichtäthers, nur quantitativ, nicht qualitativ sein. Die Mechaniker jener Zeit läugneten die Möglichkeit einer solchen Bewegung, weil sie unverträglich sei mit den hydrodynamischen Grundgleichungen, welche auf elastische Flüssigkeiten, auf Luft angewandt nur longitudinale Wellen kennen lehren. Fresnel, sich vertheidigend, machte darauf aufmerksam, dass möglicherweise in diesen Gleichungen nicht alle Kräfte berücksichtigt sein möchten, welche in elastischen Medien zur Wirkung kommen können. Er fand in der That, dass in den hydrodynamischen Gleichungen nur solche inneren Kräfte enthalten sind, welche aus einer Verdünnung oder Verdichtung des Mediums entstehen und welche wiederum eine Aenderung der Dichtigkeit hervorbringen. Er stellte sich daher die Frage, ob es in einem elastischen Medium keine anderen Kräfte gebe, ob in einem solchen System, wie es die Theilchen eines elastischen Körpers bilden, nicht auch Kräfte entstehen können aus einer Verschiebung der Theilchen, durch welche die Dichtigkeit nicht geändert wird. Wie jetzt die Sachen liegen, ist es leicht, den Standpunkt, auf den Fresnel sich stellte, klar zu machen (S. 1–2).

This account of the origin of the theory of elasticity, attributing it to the inability of the hydrodynamical equations to offer any explanation of the phenomena of light, has been accepted by several writers (see the review of our first volume in the *Bulletin des sciences mathématiques* T. 12, p. 38, 1888), but it must be distinctly borne in mind that the first propounder of the theory was Navier, an elastician of the old, or Bernoulli-Eulerian school, who both in theory and practice had frequently dealt with elastic stresses by the old methods, and whose memoir of 1827 was preceded not by optical investigations but by researches on the elasticity of rods and plates.

Neumann after briefly referring to the labours of Navier, Poisson and Cauchy concludes his first section by defining stress on their lines, i.e. by supposing inter-molecular force central and a function only of the central distance.

[1194.] The second section is entitled: *Allgemeine Lehrsätze über die Druckkräfte* (S. 8–25) and develops the usual stress equations without regard to any molecular hypothesis. The third section (S. 26–36) discusses Cauchy's and Lamé's ellipsoids of stress and the principal tractions without reference, however, to those writers: see our Arts. 610*, (iv), and 1059*. The fourth section entitled: *Das System der Dilatationen* (S. 37–51) deals with the geometry of small strains, and discusses the ellipsoids of strain and the principal stretches. The fifth section is entitled: *Beziehungen zwischen den Druckkräften und den Verrückungen* (S. 52–9). It deals only with uncrystalline and presumably homogeneous and isotropic bodies. Neumann remarks that experiment shows us that stress and strain vanish and arise coevally; hence he argues that one must be capable of being mathematically expressed as a function of the other. He then states that there can be no doubt that in uncrystalline bodies the axes of principal stretch and principal traction must coincide, and he continues:

Aus unserer Annahme, dass die Dilatationen kleine Grössen seien, folgt, dass die Druckkräfte, welche wir als Functionen jener anzusehen haben, in der Gestalt einer Entwickelung nach Potenzen der Dilatationen dargestellt werden können. Da ferner nach unserer Annahme die Dilatationen so kleine Grössen sind, dass wir nur ihre erste Potenz zu berücksichtigen brauchen, so müssen die Hauptdruckkräfte lineare Functionen der Dilatationen sein; und zwar werden sie, da sie mit jenen zugleich verschwinden, ohne Hinzufügung eines constanten Gliedes ihnen einfach proportional zu setzen sein (S. 52–3).

Obviously here Neumann falls into the same *non-sequitur* as Cauchy in his memoir of 1827 (see our Art. 614*), as Maxwell in 1850 (see our Art. 1536*), or Lamé in 1852 (see our Art. 1051*). Neumann then obtains by transformation the ordinary stress-strain relations and the body-shift equations for an isotropic elastic solid. He employs Δ for our θ, $A - B$ for our 2μ, and B for our λ. Further he uses pressures not tractions throughout his work.

The Sections 2–5 of Neumann's work form an elementary theory of elasticity, at least so far as isotropic bodies are concerned. They do not possess any particular advantages in the present state of our science.

[1195.] The sixth section of the work (S. 60–6) is entitled *Navier's Differentialgleichungen.* It deduces the body-shift equations directly by Navier's method (see our Art. 266*); this method leads to uni-constant isotropy and avoids all introduction of the stresses. In starting with Navier's investigation Neumann adopts the historical plan. He points out the objections to Navier's process (S. 66: see our Arts. 531*–2*), and then turns to Poisson's and Cauchy's treatment of the problem in his seventh section entitled: *Poisson's Ableitung der allgemeinen Gleichungen* (S. 67–79). Neumann's investigation follows fairly closely Poisson's of 1828. He deduces the shift-equations for the cases of isotropy and of three rectangular axes of elastic symmetry. The latter system he speaks of as crystalline, although it is often produced by working in bodies without crystalline structure. He says:

Zu diesen Krystallen, deren Zahl sehr gross ist, gehören alle Formen des regulären, viergliedrigen zwei- und zweigliedrigen und sechsgliedrigen Systems mit Ausnahme gewisser, hemiëdrischer Formen, bei denen die parallelen Krystallflächen fehlen, z. B. beim regulären Tetraëder. Wir nennen diese Formen die geneigtflächigen Hemiëder. Ferner findet eine solche symmetrische Vertheilung nicht mehr statt bei allen Krystallen des zwei- und eingliedrigen und des ein- und eingliedrigen Systems (S. 75).

The resulting equations involving six independent constants agree with those which would be obtained by substituting the stress-strain relations of our Art. 117 (*a*) with the rari-constant conditions $d = d'$, $e = e'$, $f = f'$, in the usual body stress-equations.

The seven sections with which we have already dealt belong to the 1857–8 notebooks. Section 8 is taken from a notebook of 1859–60, and is entitled: *Entwickelung der Gleichungen aus dem Princip der virtuellen Geschwindigkeit* (S. 80–106). This is a reproduction of the method of Carl Neumann's memoir of 1860: see our Art. 667. F. Neumann, I think, supposes the first application of the principle of virtual moments to the theory of elasticity to have been made in the above memoir, but this is hardly correct: see our Arts. 268* and 759*. The method of the *Vorlesungen* is somewhat clearer and briefer than that of C. Neumann; it is also applied to bodies with three axes of elastic symmetry.

[1196.] Section 9 (S. 107–20), taken from a notebook of 1857–8, deals with the thermo-elastic equations in the method previously adopted by Duhamel and Neumann himself. We have seen that Neumann in 1841 (see our Art. 1196*) claimed priority in the deduction of these equations, and the Editor of the *Vorlesungen* (S. vi) apparently looks upon this section as an original part of the present work. The results do not seem to be more general than those of Duhamel (1838, see our Arts. 868* and 877*) and in all cases of doubt, priority of publication must be decisive.

Neumann like Duhamel limits his equations to the range in which extension is proportional to rise in temperature. His body-stress-equations involving thermal effect (2) and (3), S. 113, are equivalent to Equations (2) of our Art. 883*; his surface stress-equations (1) and (2), S. 114, to Equation (3) of the same article; his remarks on the relations between temperature and normal pressure, and between the thermo-elastic-constant, the stretch-modulus and the thermal stretch-coefficient are equivalent to those of Duhamel in our Arts. 875* and 888*.

[1197.] § 58 (S. 115–8) is entitled: *Krystallinische Körper.* In it Neumann questions whether the thermo-elastic constant is in crystalline bodies the same for all directions. He suggests equations of the form (see our Art. 883*):

$$\rho\left(\frac{d^2u}{dt^2}-X\right)=\frac{d\widehat{xx}}{dx}+\frac{d\widehat{xy}}{dy}+\frac{d\widehat{zx}}{dz}-\beta_x\frac{dq}{dx},$$

$$\rho\left(\frac{d^2v}{dt^2}-Y\right)=\frac{d\widehat{xy}}{dx}+\frac{d\widehat{yy}}{dy}+\frac{d\widehat{yz}}{dz}-\beta_y\frac{dq}{dy},$$

$$\rho\left(\frac{d^2w}{dt^2}-Z\right)=\frac{d\widehat{zx}}{dx}+\frac{d\widehat{yz}}{dy}+\frac{d\widehat{zz}}{dz}-\beta_z\frac{dq}{dz},$$

in which he assumes, I suppose, the body to have three rectangular axes of elastic symmetry, coinciding with the thermal axes. The surface stress-equations will now be given by:

$$X'=(\widehat{xx}-\beta_x q)\cos l+\widehat{xy}\cos m+\widehat{zx}\cos n,$$

$$Y'=\widehat{xy}\cos l+(\widehat{yy}-\beta_y q)\cos m+\widehat{yz}\cos n,$$

$$Z'=\widehat{zx}\cos l+\widehat{yz}\cos m+(\widehat{zz}-\beta_z q)\cos n,$$

so that it is obvious that a rise of temperature is no longer equivalent to a uniform surface traction: see our Arts. 684–5.

Hierauf beruht die Entscheidung durch die Beobachtung. Man bestimmt durch directe Messung die Aenderung der Winkel, wenn der Druck auf die Oberfläche des Krystalls geändert wird, wenn man ihn z. B. aus dem Drucke einer Atmosphäre in den von 10 Atmosphären oder in den luftleeren Raum bringt. Auf dieselbe Weise misst man die Winkeländerung, welche durch eine Erhöhung der Temperatur, z. B. von 0° auf 100°, hervorgebracht wird. Erhält man beide Male ein entsprechendes System von Winkeländerungen, so sind alle drei Werthe von β unter sich gleich; befolgen die Aenderungen verschiedene Gesetze, so sind sie verschieden (S. 116–7).

Neumann then describes a method of making the needful measurements. He cites some experiments of Mitscherlich's (*Abhandlungen der Berliner Akademie*, 1825, S. 212) upon calcspar. This material expands in the direction of its axis owing to a rise of temperature and contracts perpendicular to the axis. The stretch for 100° C. increase of temperature was found to be ·00286 and the squeeze − ·00056. Thus the dilatation was ·00174. A similar result was exhibited by gypsum which in three different directions had different stretches or squeezes.

Neumann does not cite any experiments to determine how far the thermal results for these crystals are in accordance with those which would be produced by uniform surface tractions. He merely remarks that rods might be cut in certain directions from such crystals so that they would not change their length with change of temperature:

> Hier löst also eine krystallinische Substanz ein Problem, dessen Lösung oft sehr gewünscht wird (S. 118).

The section concludes with a paragraph deducing the amplified form of Fourier's differential equation for the conduction of heat. This is in accord with Duhamel's results cited in our Art. 883*, Equation (i).

[1198.] The tenth section of the *Vorlesungen* is entitled *Kirchhoffs allgemeine Lehrsätze* (S. 121–32). Of this section § 60 reproduces Kirchhoff's proof of the uniqueness of the solution of the equations for the equilibrium of an elastic solid: see our Art. 1255: § 61 (S. 125–8) extends the proof of the uniqueness of the solution to the case of vibrations. This, I think, had not been done by either Kirchhoff or Clebsch and is original[1] Neumann, as in the previous paragraph, supposes isotropy. We will indicate his method of proof. If there be two solutions, then their difference, given say by the shifts U, V, W, must satisfy the body- and surface-equations with abstraction of body-force and surface-load.

Consider the quadruple integral

$$\iiiint dt\,dx\,dy\,dz \left\{\left(\rho\frac{d^2U}{dt^2}+\frac{d\widehat{xx}}{dx}+\frac{d\widehat{xy}}{dy}+\frac{d\widehat{zx}}{dz}\right)\frac{dU}{dt}\right.$$
$$+\left(\rho\frac{d^2V}{dt^2}+\frac{d\widehat{xy}}{dx}+\frac{d\widehat{yy}}{dy}+\frac{d\widehat{yz}}{dz}\right)\frac{dV}{dt}$$
$$\left.+\left(\rho\frac{d^2W}{dt^2}+\frac{d\widehat{zx}}{dx}+\frac{d\widehat{yz}}{dy}+\frac{d\widehat{zz}}{dz}\right)\frac{dW}{dt}\right\},$$

which is zero owing to the body stress-equations. Integrate the stress terms by parts; the surface integrals then vanish owing to the surface stress-equations. Substitute for the stresses from the stress-strain relations, and the whole will be found a complete differential with regard to the time. Integrating out with regard to the time we find:

[1] The whole of this section is due to the lectures of 1859–60, and thus precedes Clebsch's *Treatise*. Kirchhoff's investigation was first given in the memoir of 1858: see our Art. 1255.