牛津大学研究生教材

J. G. 森姆普，G. T. 尼伯恩

代数射影几何

世界图书出版公司
www.wpcbj.com.cn

代数射影几何

J. G. 森姆普
G. T. 尼伯恩 著

图书在版编目（CIP）数据

代数射影几何 = Algebraic Projective Geometry ：英文/（英）森姆普（Semple, J. G.）著. —影印本. —北京：世界图书出版公司北京公司，2012.9
ISBN 978 - 7 - 5100 - 5284 - 2

Ⅰ. ①代… Ⅱ. ①森… Ⅲ. ①射影几何—教材—英文 Ⅳ. ①O185.1

中国版本图书馆 CIP 数据核字（2012）第 217073 号

书　　名：Algebraic Projective Geometry
作　　者：J. G. Semple, G. T. Kneebone
中 译 名：代数射影几何
责任编辑：高蓉　刘慧

出 版 者：世界图书出版公司北京公司
印 刷 者：三河市国英印务有限公司
发　　行：世界图书出版公司北京公司（北京朝内大街 137 号 100010）
联系电话：010 - 64021602，010 - 64015659
电子信箱：kjb@ wpcbj. com. cn

开　　本：24 开
印　　张：17.5
版　　次：2013 年 1 月
版权登记：图字：01 - 2012 - 6230

书　　号：978 - 7 - 5100 - 5284 - 2　　　　定　　价：69.00 元

影印版前言

自从上世纪80年代起，世界图书出版公司北京公司一直致力于与世界各国知名出版商合作，是国内最早开展购权影印图书出版工作的机构。时至今日，已经持续近30年，不仅引进的品种数独占鳌头，而且包括了大量在国际上具有深远影响的经典图书，受到了国内学者和专家的认可和好评。

现在应国内广大读者的要求，在获得牛津大学出社授权的前提下，世界图书出版公司北京公司将陆续影印出版该社各类丛书中的经典图书。牛津大学出版社是世界著名出版机构之一，每年出版的书籍、刊物超过四千种，其学术著作和教科书的作者均为相关领域的著名学者，其中不乏科学研究前沿的顶尖科学家和领军人物，书籍内容涵盖了最新的科学进展的各个方面，因此一直受到国内外科研人员和高校师生的高度评价，其中已经出版的数学和物理学系列丛书，如*Oxford Graduate Texts in Mathematics*，*Oxford Graduate Texts*，*Oxford Lecture Series in Mathematics and Its Applications*和*Oxford Mathematical Monographs*在国内有着广泛的影响，受到普遍好评。

毫无疑问，考虑到我国的国情以及科学教育发展的迫切需要，这项工作的最大受益者将是那些经济尚不富裕，但却渴望学习知识，想及时了解最新科学技术成果的国内高校和研究机构中的莘莘学子，相对原版，影印版的价格他们更容易接受。在这里，中国的读者和我们出版公司要特别感谢牛津大学出版社以传播科技知识为重，授权世界图书出版公司北京公司影印出版该社系列丛书中的部分图书。我们相信，这些图书的引进，不仅会受到数学物理等相关专业的教师和研究生的欢迎，相关领域的科研人员也将会从中受益。

前 言

为了使学生读懂这部内容较深的数学专著，本书严谨而系统地阐述了射影几何的各个方面，以期让读者在没有太大困难的情况下，掌握该领域的基本思想，并学会熟练应用。

射影几何是这样一门学科，它将重心放在代数处理方面，我们也将采取这样的思路去发展它，这样不仅是因为会为直观的几何概念和内容的严格数学论述提供一个简单的方法，而且是因为代数现在已经用在数学领域的几乎所有分支，因此假设读者已经具有一些代数方面知识是合理的。据此，我们假设读者熟悉线性代数和矩阵计算。书中只有一个例外，不再涉及纯粹的代数定理证明，这是因为该定理是我们体系的基础，而且可能在几何领域之外不会遇到和它相同的形式。在附录中，我们给出了该定理的证明。

尽管是用代数的方法研究几何，但是我们从不轻视综合方法，几何学家von Staudt，Steiner和Reye对此颇为擅长。如果一个想法在我们脑海中比别的更为显著，它就会给出体系的几何特征，并用几何的方式去思考。在我们看来，在这个传统而优美的主题中，如果仅仅是用几何语言来表述代数，则没有比这更不尽如人意的事情了。所以，我们尝试去说明尽管形式结构的基础是代数，但是结构本身却是彻底的几何，它的概念、方法和结果也都非常依赖于几何思想。

本书分为两个部分。第一部分包括两章，其中一章是历史回顾和简介，我们的目的一是和基本坐标几何衔接上，二是让读者从更高的角度去认识射影几何。第二部分阐述被重新发展了的代数射影几何理论，因此在逻辑上独立于以前的几何知识。我们相继讨论了一维、二维和三维射影空间，最后简要介绍了高维空间几何。除了讨论通常的单应性、二次曲线、二次曲面、扭曲立方和线几何之外，我们还用了相当的篇幅介绍直射变换和线性变换，因

为这些内容的重要性尤其是它们的几何方法已被人们广泛认可。纵观全书，我们非常强调欧几里德和射影结果的仿射专门化。我们两位作者以及很多授过课的人都有这样的经历，关于相当形式化的射影概念和结果的具体事例的阐述，总是会赋予它们新的盛名并激发人们的兴趣。

本书提供了大量的练习，或穿插于行文之中，或放在章节末尾。很多习题取自近年来伦敦大学优等文学士考试和数学专业理工科学士考试。还有一些习题取自剑桥数学优等考试和都柏林圣三一学院数学首次公开考试。作者感谢他们允许使用这些资源。

感谢A. E. Ingleton先生阅读了书稿并给出很多改进建议，感谢A. S. Dennis小姐对部分章节的意见。最后，感谢牛津大学出版社职员始终如一的礼待和帮助以及精美的印刷。

J. G. S.

G. T. K.

1952年4月

目 录

PART I

THE ORIGINS AND DEVELOPMENT OF GEOMETRICAL KNOWLEDGE

> 'That all our knowledge begins with experience, there is indeed no doubt . . . but although all our knowledge originates *with* experience, it does not all arise *out of* experience.'
>
> KANT: *Critique of Pure Reason*

CHAPTER I

THE CONCEPT OF GEOMETRY

OUR main purpose in this book is to construct and develop a systematic theory of projective geometry, and in order to make the system both rigorous and easily comprehensible we have chosen to build it on a purely algebraic foundation. In adopting such a course, however, we may run the risk of appearing to reduce our subject to an ingenious manipulation of symbols in accordance with certain arbitrarily prescribed rules. Although the axiomatic form is the proper one in which to present a mathematical theory, we must not lose sight of the fact that an abstract system can only be fully appreciated when seen in relation to a more concrete background; and this is the reason why we have prefaced the formal development of projective geometry with two introductory chapters of a more informal character. The present chapter is devoted to a rather general consideration of the nature of mathematics and, more specifically, of geometry, while Chapter II contains an outline of the intuitive treatment of projective geometry from which the axiomatic theory has gradually been disentangled by progressive abstraction.

The growth of geometrical knowledge in the past has been marked by a gradual shifting away from empirical observation towards rational deduction; and we shall begin by looking for a moment at this process.

Geometry is commonly regarded as having had its origins in ancient Egypt and Babylonia, where much empirical knowledge was acquired through the experience of surveyors, architects, and builders; but it was in the Greek world that this knowledge took on the characteristic form with which we are now familiar. The Greek geometers were not only interested in the facts as such, but were intensely interested in exploring the logical connexions between them. In other words, they wished to raise the status of mathematics from that of a mere catalogue to that of a deductive science—and the *Elements* of Euclid is an embodiment of this ideal. In the *Elements* we have the systematic derivation of a large body of geometrical theorems by strict deduction from a small number of axioms. The system, as is now known, is not altogether perfect, and modern mathematicians have shown how it needs to be amended; but the modifications required are comparatively slight, and there is perhaps no easier way for a student to learn to appreciate mature mathematical reasoning than by studying the first book of Euclid and observing the way in which it is constructed.

Now for the Greeks, we must remember, geometry meant study of the space of ordinary experience, and the truth of the axioms of geometry was guaranteed by appeal to self-evidence. This view persisted for a very long time, and was still accepted without question at the end of the eighteenth century—when Kant, for example, made it an integral part of his philosophy. But about that time mathematicians were already beginning to see their subject in a new light, as a branch of study not directly dependent on experience, and this change of outlook was encouraged by the discovery, early in the nineteenth century, of the non-euclidean geometries, systems consistent within themselves but incompatible with Euclid's system. Since then it has become a commonplace that the mathematician is free to study the consequences of any axioms that interest him, whether or not they have any application in experience, provided only that they are not mutually contradictory.

We see, then, that in the period which elapsed between the first beginnings of mathematics and the conscious adoption of the modern axiomatic method, two major revolutions took place in mathematical thinking. First, the mere collecting of useful or interesting facts gave place to the rational deduction of theorems; and then, much later, mathematicians began to detach themselves from experience and to concentrate on the study of *formal* axiomatic systems. Neither of the revolutions came about suddenly, and the second is in a sense still in progress.

Mathematics, as conceived today, is fundamentally the study of structure. Thus, although arithmetic is ostensibly about numbers and geometry about points and lines, the real objects of study in these branches of mathematics are the *relations* which exist between numbers and between geometrical entities. As mathematics develops, so it becomes more abstract, until at last it is seen to be concerned with networks of formal relations only, and not with any particular sets of entities between which the relations hold. The process of abstraction whereby the formal structure is by degrees detached from the concrete systems in which it is exhibited is of so great importance to the understanding of the nature of mathematics as to justify closer examination of the manner in which it takes place.

One of the simplest illustrations of the process is provided by the evolution of the concept of number. Our first rudimentary idea of number is arrived at by simple abstraction from the processes of counting and measuring ordinary objects, and this idea is adequate at the level of school arithmetic. At a more advanced stage, numbers are seen to require redefinition in purely logical terms, and several alternative definitions have, in fact, been given. In whatever way numbers are defined, however, they obey the same formal 'laws of algebra'—the associative law of addition $(a+b)+c = a+(b+c)$, the distributive law $a(b+c) = ab+ac$, etc. —and many of the standard theorems of arithmetic and algebra can be deduced directly from these laws, without any need to specify further the nature of the numbers that are represented by the symbols a, b, etc. But this is not all. When studying elementary algebra one soon becomes aware of the close analogy that exists between the algebra of polynomials and the arithmetic of whole numbers; and it is now easy to account for this analogy by pointing out that polynomials, as well as numbers, satisfy the 'laws of algebra'. This is tantamount to saying that the system of numbers and the system of polynomials have a common structure; and when once this fact is recognized it is a natural step to undertake the study of an abstract system whose nature is unspecified beyond the fact that it has this particular structure. Such a system is known in algebra as a *ring*. If, on the other hand, we apply a similar process of abstraction to the system of rational numbers or the system of rational functions, we arrive at the abstract system known as a *field*.

There is no need for recognition of structural similarity to come to an end, even at this stage. Thus we might observe, for instance, that addition of rational numbers and multiplication of non-zero rational numbers obey similar laws; and we could then verify that the additive structure of a field and its multiplicative structure (when the element zero is excluded) are formally alike. Carrying the process of abstraction one stage farther, we could now introduce the abstract system known as a *group*.

Mathematics, then, is concerned with abstract systems of various kinds, each defined by a suitable set of axioms, which serves to characterize its structure. But although, from the point of view of pure mathematics, each structure is regarded as self-contained, the mathematical scheme usually has one or more concrete *realizations*; that is to say, the structure is usually to be found (possibly only to a certain degree of approximation) in a more concrete system. Abstract euclidean geometry of three dimensions, for instance, has as one of its realizations the structure of ordinary space. Indeed this is what led to its discovery, as well as what makes it so much more interesting than other systems which are logically of equal status with it. We do not, of course, always have to go all the way back to everyday experience for a realization of a mathematical formalism, since one is usually provided, as in the arithmetical example already considered, by a more concrete part of mathematics itself. One of the most important instances is the widespread occurrence of the group structure, which is found not only in additive and multiplicative groups of numbers, but also in groups of transformations and groups of matrices. Since this type of structure pervades much of mathematics, we may say that it is especially *significant*.

In this book we shall study the structure of projective geometry which, as is well known, is closely associated with certain simple algebraic structures, and with linear algebra particularly. Since the relevant algebra is part of every mathematician's essential equipment, we shall take it for granted that the reader is already familiar with it.

What we have said so far about the nature of mathematics holds quite generally, but when we limit the discussion to geometry we are able to be rather more specific. The structures studied in this branch of mathematics occur in experience as spatial structures, and from this alone we can infer something of their general character.

If, in fact, we turn back once again to Greek geometry, we may recall that the geometrical knowledge with which the Greeks began was derived ultimately from measurements made upon rigid bodies, and was therefore essentially a knowledge of shapes. Now the shape of a body can be conceived as determined by those relations between its parts which remain unaltered when the body is moved about in space. Whenever one body can be made in this way to take the place of another, the two bodies have the same shape; and they are then equivalent as regards their geometrical properties, or, in the language of elementary geometry, 'equal in all respects'. It will be remembered that in order to prove that certain sets of conditions are sufficient to ensure the congruence of two triangles Euclid showed that, if the conditions are satisfied, one triangle may be placed so as to bring it into coincidence with the other.

The idea of studying those properties of bodies which remain unaltered when the bodies are displaced in any way is most suggestive to a modern mathematician. In the language now in use, we would say that the geometrical (or, more accurately, the euclidean) properties of a body are invariant with respect to the operation of displacement in space; and invariance with respect to a certain *kind* of operation at once suggests the existence of an underlying *group* of operations. In the present instance the appropriate group is not far to seek. The totality of all displacements in space is a group of transformations; two bodies are congruent if and only if one can be made to take the place of the other by an operation of the group; and the shape of a body is determined by those of its spatial characteristics which are invariant with respect to the whole group. This, then, is the nature of euclidean geometry—it is the invariant-theory of the group of displacements.

Euclidean geometry, however, is not the whole of geometry. Early in the nineteenth century it was realized that other systematic collections of geometrical properties are possible besides that of Euclid, and in 1822 Poncelet published his *Traité des propriétés projectives des figures*, the first systematic treatise on projective geometry. In constructing this system Poncelet was fully conscious that his classification of geometrical theorems was based upon a new kind of fundamental operation, namely conical projection. A projective property of a figure is, in fact, simply a property that is invariant with respect to projection, and this enables us easily to identify the associated group of transformations. Confining ourselves, for simplicity, to two-dimensional geometry, we may consider the totality of all those transformations of the plane into itself which can be resolved into finite chains of projections from one plane on to another; and it is clear that this totality of transformations is a group and that it has plane projective geometry as its invariant-theory. Since the euclidean group, consisting of all displacements of the plane, may be shown to be a proper subgroup of the projective group, it follows at once that every projectively invariant property is also a euclidean invariant, whereas not every euclidean property is projective.

If we were now to take any arbitrarily chosen group of transformations of the plane into itself (containing the group of displacements as a subgroup) we could use this group in order to *define* an associated system of geometry; and all such systems are, mathematically speaking, of equal status. This was the general principle laid down by Klein in his famous *Erlangen Programme* of 1872.† Some of the geometries that can be obtained in this way, such as euclidean geometry, affine geometry, and projective geometry, are very well known; others, such as inversive geometry (which arises from the group of all transformations that can be resolved into finite sequences of inversions with respect to circles) are known but not usually studied in much detail; and yet others are presumably ignored altogether.

We shall confine our attention to the three geometries first mentioned—the geometries of the *projective hierarchy*—and since this restriction is somewhat arbitrary from a purely mathematical point of view, we should perhaps give some indication of why we choose to impose it. In the first place, euclidean geometry is of particular interest on account of its close connexion with the space of common experience, and this alone is sufficient to single it out for special attention. It so happens, however, that euclidean geometry is complicated; and we can appreciate it better when we relate it to projective geometry, where the structure is very much simpler. Projective geometry is more symmetrical than euclidean, by virtue both of the existence of a principle of duality and also of the fact that it may be handled by means of homogeneous coordinates. When homogeneous coordinates are used

† Klein: *Vergleichende Betrachtungen über neuere geometrische Forschungen* (Erlangen, 1872). Reprinted in *Mathematische Annalen*, **43** (1893).

for this purpose, the algebra has the merit of being either already linear or else readily made so. Thus the system of projective geometry is easy to work out and equally easy to comprehend when it has been worked out. Furthermore, projective transformations have the property of transforming conics into conics; and this means that the conic takes its place as naturally in projective geometry as does the circle in euclidean geometry. Finally, the essentials of euclidean geometry may be treated projectively by the simple artifice of introducing the line at infinity and the circular points. We thus have two geometries, projective geometry and euclidean geometry, which fit naturally together and which between them include most of the classical geometrical theorems. It is convenient to take in conjunction with them affine geometry, an intermediate geometry that is more general than euclidean but less so than projective; and the projective hierarchy is then complete.

What has been said so far concerns the subject-matter of our book, and it still remains for us to say something of the kind of approach that we shall use. It is customary to distinguish between two modes of reasoning in geometry, commonly referred to as synthetic and analytical. In a synthetic treatment we argue directly about geometrical entities (points, lines, etc.) and geometrical relations between them, whereas in an analytical treatment we first represent the geometrical entities by coordinates or equations, in order to be able to use the technique of algebraic manipulation. Since the discussion of projective geometry which follows in Part II is to be analytical, we shall conclude this chapter by touching upon the use of coordinates; but it should be realized, nevertheless, that we are under no logical compulsion to introduce a coordinate system at all. In the *Elements*, as in all Greek treatises, euclidean geometry is treated synthetically, and synthetic treatments of projective geometry are to be found in a number of modern books on the subject.†

Coordinates were first introduced into geometry by Descartes, in the seventeenth century, and the fruitfulness of the innovation soon became apparent. The older method of labelling figures was by letters of the alphabet, as in 'the triangle ABC', but such labels

† The first work of this kind was von Staudt's *Geometrie der Lage* (Nuremberg, 1847). A standard text-book, written in a similar spirit, is Veblen and Young's *Projective Geometry* (Boston, 1910).

were in fact no more than arbitrarily assigned names. Descartes's new technique of coordinates, on the other hand, made use of a system of labels which itself possesses a mathematical structure capable of reflecting the structure of the system labelled. This method of labelling has since become indispensable in mathematics, and the domain in which it can be applied now extends far beyond that originally envisaged by Descartes. In geometry itself, not only points but also lines and other entities can be represented by sets of coordinates; and in dynamics—to take an instance of another kind—the configuration of a system is ordinarily specified by n coordinates $q_1, q_2, \ldots, q_n$.

We have now seen how mathematics may be looked upon as a study of formal structure, and how geometry may be fitted into the general scheme. What has been said so far has been of a rather general character, and we must now turn more specifically to the details of the geometries of the projective hierarchy. This will be the topic of the second chapter of Part I, in which our purpose will be to recall enough of the elementary treatment of projective geometry to enable the reader to appreciate the process of abstraction which leads to the formal system of Part II.

CHAPTER II

THE ANALYTICAL TREATMENT OF GEOMETRY

THIS chapter is devoted, for the most part, to a discussion of the basic ideas involved in projective geometry and the apparatus of coordinates which allows them to be handled algebraically, and the point of view adopted is essentially elementary. The whole account is to be regarded as introductory, and in Part II a completely fresh beginning will be made. The formal system to be presented there is wholly abstract and independent of all previous geometrical knowledge; but even so, an elementary treatment such as that given in the present chapter is necessary as a psychological though not a logical presupposition of the more advanced theory. It alone can give body to the abstract formalism.

This chapter is not meant to be more than a summary, and the reader who desires a fuller account of the subjects touched upon in it is referred to Graustein: *Introduction to Higher Geometry* (New York, 1930).

§ 1. THE PROJECTIVE HIERARCHY

We have already referred in Chapter I to the three geometries of the projective hierarchy and the possibility of defining them in terms of certain groups of transformations. It will be convenient, before proceeding further, to make these ideas more precise by giving a few details of each of the geometries; and once again we shall confine ourselves to the geometry of the (real) plane.

Euclidean geometry

The underlying group (p. 5) is the group of all displacements in the plane. The simplest invariant of this group is *length*, or the distance between two points. *Angle* is another invariant, and it follows from a theorem on congruent triangles (Euclid, I. 4) that angles may be characterized by suitably chosen lengths.

Among the figures appropriately studied in euclidean geometry is the *circle*, or locus of a variable point whose distance from a fixed point is constant. The theorems which properly belong to euclidean geometry include most of those in the *Elements*.

Analytically, euclidean geometry is best handled by means of rectangular cartesian coordinates, since, by virtue of the theorem of Pythagoras, the expression for the distance between two points then has a particularly simple form. Euclidean geometry may also be handled by vectors, the length of a vector being expressed in terms of the scalar product.

Projective geometry

The underlying group consists of all finite chains of projections that begin and end on the given plane. Relations of *incidence*, *collinearity*, and *tangency* are all projectively invariant, and *cross ratio* (cf. p. 17) is an invariant quantity.

A figure that is appropriately studied in projective geometry is the conic, since every conic is obtainable by projection from a circle.

Analytically, projective geometry is best handled by means of projective coordinates, which will be defined in § 5. These coordinates are expressible in terms of cross ratios. Vectors, as ordinarily defined in elementary books, have no application in projective geometry proper.

Affine geometry

Affine geometry occupies an intermediate position between euclidean geometry and projective geometry. The underlying group is generated by all *parallel* projections in space. The simplest invariant quantity for this group is the *position ratio* AP/PB of a point P with respect to two points A, B with which it is collinear. All projective properties are *a fortiori* affine properties; and when we pass from the projective group to the more restricted affine group, *parallelism* is introduced as a new invariant property.

Among the figures entering appropriately into affine geometry are the *parallelogram* and the separate kinds of conic, the *ellipse*, *hyperbola*, and *parabola*. The theorems which belong to affine geometry include the theorem on the concurrence of the medians of a triangle, Ceva's theorem, and the theorems on diameters of conics.

The coordinates which are suitable for handling affine geometry are oblique cartesian coordinates (perpendicularity of the axes in this case producing no essential simplification) or areal coordinates (see p. 25 below). Vectors may also be used; and since the scalar product is not involved, only linear vector algebra is required.