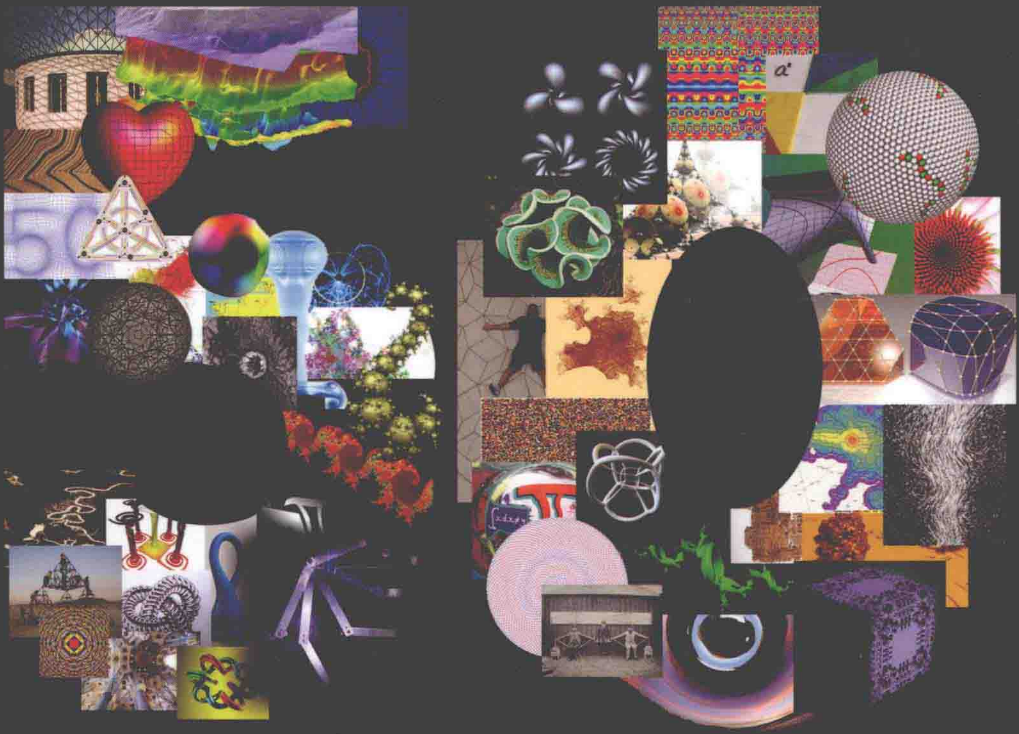
WITH A FOREWORD BY DARA O BRIAIN



VISIONS OF MATHEMATICS

EDITED BY SAM PARC

OXFORD

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SAM PARC
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FOREWORD: A BAD MATHEMATICIAN'S APOLOGY

DARA O BRIAIN

Whenever an anthology of great mathematics is compiled, the foreword, the warm-up man, is the best position I'll get on the bill. And this is for good reason. I started on the same path as these people, who will be here in a page or two with their bright ideas and their elegant proofs. In fact, if multiverse theory is to be believed there may even be a region of spacetime in which I continued there, struggling to keep up. This is our reality, though, and in this one, they kept going with the equations and I ran off to the circus. I left a field where we compare the properties of one thing to another in the hope of furthering our knowledge of both, and joined a trade where we say things like 'Sex is a lot like a bank account ...'

This should never be construed as maths' loss for comedy's (arguable) gain.

Studying the mathematical sciences requires not just a quick mind, but a dedicated one. An entire syllabus can often be series of linked statements rolling on logically from the very first. If you miss a lecture, perhaps because college was also where you discovered telling jokes to crowds, partying, and the sort of exotic women who weren't in your secondary school, you can find yourself playing a year-long game of catch-up. You still attend the lectures but it becomes an exercise in the transcription of mathematics rather than the comprehension of it.

In the only facet of my life at the time which could have been described as monastic, I would sit at a desk in my lectures, scratching out my personal copy of the great theorems as the lecturer argued them out for me on the blackboard. Each lecture would cover about six A4 pages, furiously scribbled in the hope of being re-read at some later date when it might make sense in context. The high-water mark of this was a proof in second-year group theory for something called *Sylow's theorems*. The details of what Sylow theorised needn't concern you now. What is important was that it took four one-hour lectures to explain/dictate what is essentially a single result, scribbled furiously over 27 sheets of A4 paper. I looked it up online a minute ago and could make no sense of it there either.

I recently downloaded an Oxford University course on quantum mechanics. 27 lectures, recorded 'live', covering the sort of topics I had only seen before through a blur of a newly discovered social life. Here was my chance to put right the mistakes of a younger man, sat in my home with a notepad and pen.

I pressed 'play' on the first one, the lecturer began, and immediately I had to start writing frantically as the blackboard was filled with line after line of symbols. For Proust, nostalgia was triggered by the taste of madeleine cake; for me, it was scribbling maths furiously and trying to keep up with a sustained argument. Luckily, just when I could feel it slipping away from me, the cat walked across the keyboard, stood on the space bar, and the video stopped. I was suddenly back to being a 41-year-old man, sitting in his kitchen on the family computer, but almost out of breath from the writing.

By contrast, a friend sneaked me into a third-year philosophy lecture once. Pen at the ready, notepad open, I found myself attending what I could only describe as 'a chat'. A man burbled amiably at the head of the class and the students nodded, deigning occasionally to jot the odd note down. The lecture was about the veil of perception, I recall, the theory that what we 'sense' about the reality around us is only what is fed through our physical senses and since this 'veil' exists between us and presumed, actual reality, we cannot come to any definite conclusions about the nature of that reality.

I might have that wrong. I almost definitely have some part of it incorrect. I'd even welcome it being corrected by any passing philosophy professor. The point is, though, that's what I remember from a single lecture 19 years ago. I did four lectures about Sylow's theorems, plus surrounding material and question sheets, and then an exam, which I presumably passed. And I still had to google it this evening to remind myself what it was about.

So, maths is intense. But you stick with that because it is so beautiful. And this beauty is often not clear to outsiders.

You'll often hear mathematicians talk about beauty. Usually it is in reference to an elegance or wit in the argument; an efficiency in paring a question down to its most important parts; how a few lines of simple logic reveal a deep truth about numbers, or shape, or symmetry.

There are other things that make maths beautiful, however, particularly in comparison to the other sciences. For a start, it is delightfully uncluttered with Latin names. Not for maths the waste of energy learning off genus, phylum, and kingdom of a snail that looks pretty much the same as the next snail, except for a slightly stripier shell. Or the proper name of that tube that connects the inner ear of a horse to that other part of the inner ear of the horse.

Equally, maths was never too bothered about measuring things. I spent four years doing maths and probably never saw a number, other than the one next to the question, or at the bottom of the page. We'll do the variables and somebody with an oscilloscope can plug in the numbers. Maths will predict the results and somebody else can go off and build a 17-mile-long tunnel under Switzerland and tell us if we're right. This is not just disdain; there's an admission of practical ineptitude implied too. There are 500,000 rivets in the Large Hadron Collider. If they'd left them to the mathematicians those packets of protons would have gone four feet before hitting a stray lump of solder. Or a cufflink. Or a mathematician soldered to the tube, by his cufflink.

I have become publicly associated with astronomy, for example, by virtue of a television show I do. However, I know I would be of no use as an actual astronomer, a field built on meticulous observation and measurement: tracing points of lights across the sky, night after night, in the hope of spotting the tiniest aberrations, revealing the hugest things. Or you could write down the equations of cosmology and let somebody else do all that. That's the beauty of maths.

Sometimes beauty is not enough, though.

I visited my head of department at Mathematical Physics at the end of my degree to drop the bombshell that I wouldn't be seeking a Masters place; this was so that he could politely feign surprise and disappointment. 'We'll miss you,' he said, rising to the challenge. 'You had a tremendous … flair … in your mathematics.' It was only after I'd left the room that I realised that this was not intended as a compliment. They had been looking for even a trace of rigour.

The pieces in this collection have plenty of flair. They sparkle with bright ideas and clever results. They're best read with a pen and paper so that you can try out some of the leaps in imagination yourself. There will often be the surprising results of comparing the properties of one thing to another. There may even be the odd good joke.

And if you ever feel intimidated during these pages, as some people do with maths, let me offer a reassurance.

At some point during their college career, each fine mind whose work is contained within this book was just another student furiously transcribing superscript *i*'s and subscript *j*'s, and hoping that they would get a moment's pause to catch their breath and pick the misplaced solder from their hands.

PREFACE

There is an old saying, that God made the whole numbers and that all of the rest of mathematics is the invention of man.* If that is true, what was going on when fifty was made? Because, from a human perspective, fifty seems to have a particular significance. We celebrate fiftieth birthdays, wedding anniversaries, and any other commemorations you might care to mention, with special verve. Fifty also represents a coming of age for many of us personally, an age where we might want to take on a fresh challenge, embrace new visions of life. The phrase 'fifty-fifty' implies a perfect degree of fairness. And fifty per cent is halfway there – a sense of achievement, yet hinting at more to come.

This book celebrates not just the number 50 and the concept of fiftiness, but the whole of the rest of mathematics, as well as the people involved in its creation, and its underpinning of many things we take for granted. It pays tribute to the relevance that mathematics has to all of our lives and embraces an exciting vision of the next 50 years of mathematics and its applications.

Fifty is of course also a number, and as a number it has its own special properties. To a mathematician, 50 is the smallest number that is the sum of two non-zero square numbers in two distinct ways, $50 = 1^2 + 7^2 = 5^2 + 5^2$; it is the sum of three successive square numbers, $50 = 3^2 + 4^2 + 5^2$. Moreover, to chemists it is a magic number, as atomic nuclei with 50 nucleons are especially stable.

The original motivation for this book was the 50th anniversary of the UK's Institute of Mathematics and its Applications (IMA) in 2014, but the project grew in scope. Moreover, it would seem that, during the last 50 years, mathematics has come of age. School mathematics has undergone a transition from arithmetic, algebra, and geometry alone to sets, topology, computer imagery, and an appreciation of the importance of mathematics in its many diverse applications. We have seen the birth of new scientific paradigms, for example in chaos theory, string theory, and genomics. We have experienced the computer revolution: nearly every home now has a computer with more calculating power than all of the world's computers combined 50 years ago. We have the new mathematics of the digital age, where massive datasets are pervasive and the mathematics-based technologies of the Internet have completely transformed our lives. We are also in the age of computer modelling: the entire human physiome and the climate of the Earth over millennia can now be simulated using billions of variables. Given this progress in the last 50 years, we must ask with a sense of wonder what the future of mathematics and its applications will be.

The main content of this book is a collection of 50 original essays contributed by a wide variety of authors. The topics covered are deliberately diverse and involve concepts from simple numerology to the very cutting edge of mathematics research. Each article is designed to be read in one sitting and to be accessible to a general audience. Nevertheless, I have not asked the contributors to shy away from using equations and mathematical notation where necessary. For those of little or no mathematical training, please don't worry; it should be possible to get the sense of any article without reading the symbols.

*This is attributed to the 19th-century Prussian mathematician Leopold Kronecker.

Contributors were chosen in a rather haphazard way. Aided by my editorial team (see below), we brainstormed a list of potential leading mathematicians who would like to share their vision of mathematics with us. We were delighted and humbled by just how many agreed, which left us with a difficult problem of selecting the final 50. In a few cases the piece is based on something that has appeared elsewhere in another form, notably in the IMA's *Mathematics Today* magazine or in the truly excellent *Plus Magazine* online (<http://plus.maths.org>). We also ran a competition in both *Mathematics Today* and *Plus* to invite new authors to contribute to this project. You will find a small selection of the best entries we received sprinkled among those from more established authors.

The essays are reproduced in alphabetical author order, which makes a somewhat random order thematically. Nevertheless, the contributors were asked for pieces that fall into one of five broadly defined categories: mathematics or mathematicians from the last 50 years (a time frame that is somewhat stretched in a couple of cases); quirky mathematics; mathematics of recreation; mathematics at work; and the philosophy or pedagogy of mathematics. Each piece is intended as a mere appetiser and, where appropriate, concludes with a few items of suggested further reading for those tempted into further delights.

There is also other content. There are 50 pictorial 'visions of mathematics', which were supplied in response to an open call for contributions from IMA members, *Plus* readers, and the worldwide mathematics community. Again, these images are presented in no particular order; nor are they necessarily supposed to be the top 50 images of mathematics in any objective sense. Mathematics is a highly visual subject, and these images are there to illustrate the breadth of that vision. An attribution and a very short description of each image are supplied at the back of the book. I have also been tempted to include other mathematical goodies related to the number 50. In particular, as I said at the start, $50 = 3^2 + 4^2 + 5^2$. It might not have escaped your attention that, in addition, $3^2 + 4^2 = 5^2$. This makes (3; 4; 5) the first *Pythagorean triple*, the smallest whole numbers for which there is a right-angled triangle with these side lengths: 3, 4, and 5. This (admittedly slender) link between 50 and Pythagoras's theorem allows me to introduce a running theme through the book, presented in three chunks – a series of 'proofs' of this immortal theorem in a number of different literary styles. I hope that you (and Raymond Queneau, from whose work my inspiration came) will forgive me for this.

There are many people who deserve thanks for their involvement in putting this book together. First and foremost, I should like to thank the IMA – particularly David Youdan, Robert MacKay, John Meeson, and members of the IMA council – for keeping the faith when what was originally envisaged as a small pamphlet to accompany their 50th birthday celebrations grew and GREW. All the profits from the sale of this book will go to support the current and future work of the IMA in promoting and professionalising mathematics and its applications.

I am deeply indebted to all the contributors to this book, who have fully embraced the spirit of the project, and have given their work for free, agreeing to waive any royalties. Also, particular thanks go to Steve Humble, Ahmer Wadee, Marianne Freiberger, Paul Glendinning, Alan Champneys, Rachel Thomas, and Chris Budd, the informal editorial committee of IMA members who have aided me in all stages of this project, from suggesting contributors and liaising with them, to editing the copy. Keith Mansfield at OUP has been invaluable and has gone way beyond what one might normally expect of a publishing editor. I should like to single out Maurice MacSweeney, without whose forensic skills key parts of the book's manuscript might have been lost forever. I should acknowledge Nicolas Bourbaki, whose style of writing has greatly influenced the way I work. And, finally, thanks go to my family and to Benji the dog, who have had to suffer fifty forms of hell as I have juggled the requirements of my many other commitments.

CONTENTS

CHAPTER 1

What's the problem with mathematics?

DAVID ACHESON

Why do so many people have some kind of problem with mathematics?

The real truth, so far as I can see, is that most of them are never let anywhere near it. They see mathematics as being about aimless calculations, rather than about discovery and adventure. In particular, they see none of the surprise that often comes with mathematics at its best. Yet I had my first big mathematical surprise at the age of just 10, in 1956. I was keen on conjuring at the time, and came across the following mind-reading trick in a book – see Fig. 1.1.

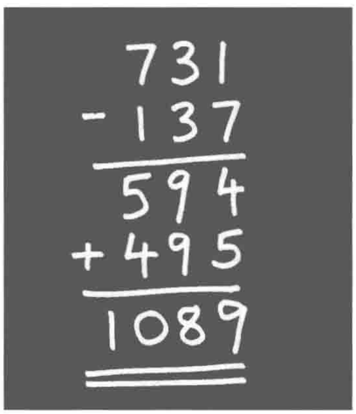

Fig 1.1 The 1089 trick.

Think of a three-figure number. Any such number will do, as long as the first and last figures differ by 2 or more. Now reverse your number, and subtract the smaller three-figure number from the larger. Finally, reverse the result of that calculation and add. Then the final answer will always be 1089, no matter which number you start with! And while it may not be very serious mathematics, I have to tell you this: if you first see it as a 10-year-old boy in 1956, it blows your socks off (see Chapter 34 for an even more subtle number puzzle).

Start with geometry?

Over many years now I have tried to share my sense of wonderment with mathematics through so-called family or community lectures in schools, usually held in the evening. The age range at such events can be enormous, from grandparents to very young children indeed. And all you can really assume is that each family group has at least one person who is good at sums.

Now the fastest way I know of introducing the whole spirit of mathematics at its best, especially to a very young person, is through geometry. And, as it happens, I once had the opportunity to try this out, with the help of a teacher, on a small group of 8-year-olds in a primary school in Hertfordshire. We started with parallel lines, and used these to prove that the angles of a triangle always add up to 180°. You can see this by staring hard at Fig. 1.2.

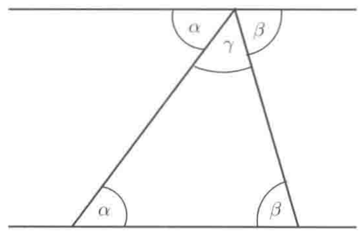

Fig 1.2 A geometric proof that the angles in a triangle sum to 180°

Next, we noted that the base angles of an isosceles triangle are equal, and agreed that this was fairly obvious. (The clock was ticking.) Then they all did some practical experimentation, finding that the angle in a semicircle always seems to be 90°. This caused a real stir, and one of them even shrieked in amazement. Finally, we used our two results on triangles to prove this, as reproduced in Fig. 1.3.

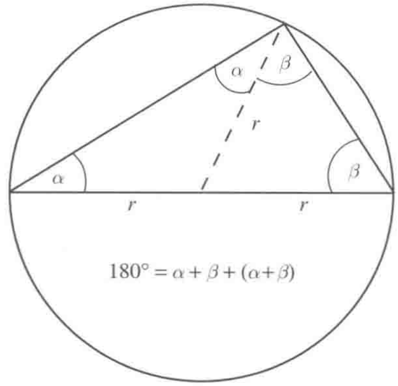

Fig 1.3 Proof that the angle inscribed in a semicircle is 90°.

As I recall, we made this particular journey – from almost nothing to what is arguably the first great theorem in geometry – in just half an hour or so. And nobody burst into tears.

Proof by pizza

With a slightly older group, the whole idea of an infinite series, such as

$$1/4 + 1/16 + 1/64 + \ldots = 1/3,$$

offers some real possibilities. Many people are genuinely surprised that such a series can have a finite sum at all. And, in my experience, they can be even more struck by the elegance of an off-beat derivation of this result, which I call *proof by pizza*.

Take a square pizza, of side length 1 unit (a foot, say), which therefore has area 1. Cut it into four equal pieces, and arrange three of these in a column, as in Fig. 1.4. Then cut the piece that is left over into four equal pieces, and arrange three of those, likewise, in a column as in panel (b). Now keep on doing this for ever. In this way (if you ignore bits of cheese falling off, etc.), you generate three identical rows. Each row is, in terms of area, equivalent to the original infinite series. The total area must still be 1, so each series must add up to 1/3. Then you eat the pizza (c), and that completes the proof.

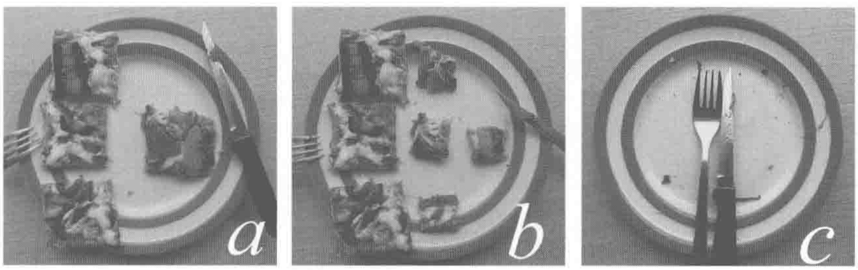

Fig 1.4 Proof by pizza.

It is then, arguably, a smallish step to present – albeit without proof – one of the more subtle pleasures of mathematics at its best, namely *unexpected connections* between different parts of the subject. For this purpose, the so-called Gregory–Leibniz series (which was actually first discovered in India) shown in Fig. 1.5 is, in my view, hard to beat. After all, everybody knows what an odd number is, and everybody knows that π is all about circles. But why should these two ideas be related at all, let alone in this beautifully simple way?

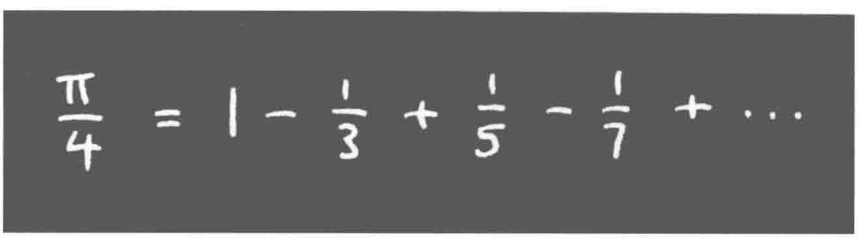

Fig 1.5 The Gregory–Leibniz series.

Not quite the Indian rope trick

One of the major functions of mathematics, surely, is to help us understand the way the world works, and, in particular, to get where physical intuition cannot reach. I like to share with audiences my most memorable experience of this, which came one wet, windy afternoon in November 1992. For some weeks before, strange things had been happening in my computer models of multiple pendulums, so I finally sat down with a blank sheet of paper and tried to find, and prove, a general theorem.

And just 45 minutes later, against all my intuition, it dropped out, and implied that a whole chain of N linked pendulums can be stabilised *upside down*, defying gravity – a little bit like the Indian rope trick – provided that the pivot at the bottom is vibrated up and down by a small enough amount and at a high enough frequency. You can see a picture of the pendulums in Fig. 1.6 on the board just behind the guitar-playing penguin, who I think is supposed to be me.

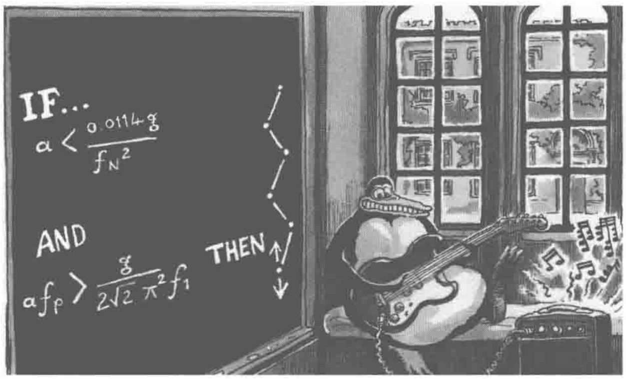

Fig 1.6 Steve Bell's interpretation of my room in Oxford.

And when my colleague Tom Mullin verified these predictions experimentally, the whole business rather captured the public imagination and eventually featured in newspapers and on national television. Now, the theorem is too quirky, I think, to be of any great significance for the future of the world, but it was still the most exciting 45 minutes of my mathematical life, and whenever I talk of that afternoon to young people I like to think that it spurs them on a bit.

Real or imaginary?

A few years ago, I wrote a book on mathematics for the general public, and the final chapter – on so-called imaginary numbers – started with the cartoon in Fig. 1.7. The equation on the television screen, due to Euler, is one of the most famous in the whole of mathematics, for it provides an extraordinary connection between π, the number $e = 2.7182818\ldots$ (the base of natural logarithms) and i, which is the 'simplest' imaginary number of all, namely the square root of -1.

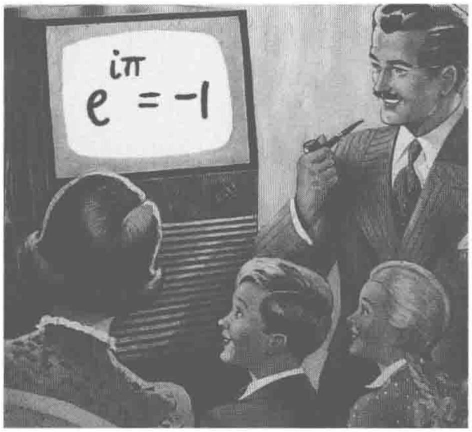

Fig 1.7 Euler's formula.

With the television cartoon, I am effectively asking whether the above scene might really occur at some point in the future. Not perhaps with that particular family, or even with that particular television, but . . .

Some will say that this is hoping for too much, and I suspect the Oxford mathematician Edward Titchmarsh would have agreed, because he wrote, in his 1959 classic *Mathematics for the general reader,* 'I met a man once who told me that, far from believing in the square root of minus one, he didn't even believe in minus one.'

But I am an optimist. On one occasion, during one of my community lectures at a school in North London, I was midway through proof by pizza when I happened to notice a particular little boy, aged about 10, in the audience. And a split second after delivering the punchline of my proof, when a deep idea suddenly becomes almost obvious, I actually saw the 'light bulb' go on in his head, and he got so excited that he fell off his chair.

And, in a sense, that fleeting moment says it all.

FURTHER READING

[1] David Acheson (2010). *1089 and all that: A journey into mathematics.* Oxford University Press (paperback edition).

[2] David Acheson and Tom Mullin (1993). Upside-down pendulums. *Nature*, vol. 366, pp. 215–216.

[3] Edward Titchmarsh (1981). *Mathematics for the general reader.* Dover Publications.

CHAPTER 2

The mathematics of messages

ALAN J. AW

We all write messages, be they SMSs to our loved ones, emails to friends, or even telegrams. Yet surprisingly few of us ask, 'How are these messages or data, which are stored in the deep recesses of iCloud, the Internet repository, or some obscure "geek facility", transmitted with such clarity and speed?' Perhaps with a little help from physics, we could surmise that messages are transmitted by waves. However, this alone does not explain how the data are transmitted with high accuracy. In fact, we intuit that waves travelling along a non-vacuum medium are most likely to experience disturbances, or *perturbations*, which would introduce errors into the data being transmitted. Moreover, these perturbations are likely to be irreversible, i.e. the waves do not undergo a self-correcting mechanism. So, the principal question to ask is: how could data still be transmitted with such high fidelity? Or, in more pragmatic language, how is it that we enjoy such speedy and accurate means of communications every day?

Messages, information, and data

It turns out that there is an underlying mathematical theory behind all these data or *information* transfers. The transmission of data via waves from the Internet to our smart devices is a specific example of a more general and abstract notion of data transfer from one point to another. Here, a point is either a sender (i.e. an information source) or a receiver (i.e. a destination), and could be, for example, a satellite or a mobile phone.

In this general model of information transfer the sender first sends some information or message to an *encoder*, which then encodes it by simply representing the message using a suitable mathematical structure. A historically significant example is the use of binary digits or *bits*, i.e. zeros and ones, to encode black and white pictures. In this encoding technique, which was implemented by NASA in the 1960s, the picture was divided into equally sized boxes so that each box was either fully black or white; and the encoder used the digit 1 to represent every black box and 0 for every white box, effectively giving rise to an array of 1s and 0s (or, in mathematical parlance, an *incidence matrix*).

The next step after encoding is to communicate the encoded information, which we refer to as *data*, to a receiver. In doing so, the data are transmitted across a medium or *channel* – in the case of the satellite it would be the atmosphere and the areas of the galaxy near the surface of the