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85

THE GEOMETRY OF FRACTAL SETS

K. J. FALCONER

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This book contains a rigorous mathematical treatment of the geometrical aspects of sets of both integral and fractional Hausdorff dimension. Questions of local density and the existence of tangents of such sets are studied, as well as the dimensional properties of their projections in various directions. In the case of sets of integral dimension the dramatic differences between regular 'curve-like' sets and irregular 'dust-like' sets are exhibited. The theory is related by duality to Kayeka sets (sets of zero area containing lines in every direction). The final chapter includes diverse examples of sets to which the general theory is applicable: discussions of curves of fractional dimension, self-similar sets, strange attractors, and examples from number theory, convexity and so on. There is an emphasis on the basic tools of the subject such as the Vitali covering lemma, net measures and Fourier transform methods. The author has brought together much of this area of geometric measure theory, which has previously only been available in technical papers. Some of the proofs have been simplified, new material is included and further developments are surveyed. Exercises are included at the end of each chapter.

This book will be of interest to research mathematicians and graduate students in mathematics concerned with analysis and geometry. It will also provide a useful account for those meeting sets of fractional dimension in, for example, number theory, non-linear differential equations, turbulence, Brownian motion or astronomy.

A reviewer's comment:
'This book is filled with the geometric jewels of fractional and integral Hausdorff dimension. It contains a much-needed unified notation and includes many recent results with simplified proofs. The theory is classically and rigorously presented with applications only alluded to in the introduction. Each chapter contains a short and important problem set. This is a lovely introduction to the mathematics of fractal sets for the pure mathematician.'
American Mathematical Monthly

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FALCONER: THE GEOMETRY OF FRACTAL SETS

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The geometry of fractal sets

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85 *The geometry of fractal sets*

This book is dedicated in affectionate memory of my Mother and Father

Preface

This tract provides a rigorous self-contained account of the mathematics of sets of fractional and integral Hausdorff dimension. It is primarily concerned with geometric theory rather than with applications. Much of the contents could hitherto be found only in original mathematical papers, many of which are highly technical and confusing and use archaic notation. In writing this book I hope to make this material more readily accessible and also to provide a useful and precise account for those using fractal sets.

Whilst the book is written primarily for the pure mathematician, I hope that it will be of use to several kinds of more or less sophisticated and demanding reader. At the most basic level, the book may be used as a reference by those meeting fractals in other mathematical or scientific disciplines. The main theorems and corollaries, read in conjunction with the basic definitions, give precise statements of properties that have been rigorously established.

To get a broad overview of the subject, or perhaps for a first reading, it would be possible to follow the basic commentary together with the statements of the results but to omit the detailed proofs. The non-specialist mathematician might also omit the details of Section 1.1 which establishes the properties of general measures from a technical viewpoint.

A full appreciation of the details requires a working knowledge of elementary mathematical analysis and general topology. There is no doubt that some of the proofs central to the development are hard and quite lengthy, but it is well worth mastering them in order to obtain a full insight into the beauty and ingenuity of the mathematics involved.

There is an emphasis on the basic tools of the subject such as the Vitali covering theorem, net measures, and potential theoretic methods.

The properties of measures and Hausdorff measures that we require are established in the first two sections of Chapter 1. Throughout the book the emphasis is on the use of measures in their own right for estimating the size of sets, rather than as a step in defining the integral. Integration is used only as a convenient tool in the later chapters; in the main an intuitive idea of integration should be found perfectly adequate.

Inevitably a compromise has been made on the level of generality adopted. We work in n-dimensional Euclidean space, though many of the ideas apply equally to more general metric spaces. In some cases, where the proofs of higher-dimensional analogues are much more complicated, theorems are only proved in two dimensions, and references are supplied for the extensions. Similarly, one- or two-dimensional proofs are sometimes given if they contain the essential ideas of the general case, but permit simplifications in notation to be made. We also restrict attention to Hausdorff measures corresponding to a numerical dimension s, rather than to an arbitrary function.

A number of the proofs have been somewhat simplified from their original form. Further shortenings would undoubtedly be possible, but the author's desire for perfection has had to be offset by the requirement to finish the book in a finite time!

Although the tract is essentially self-contained, variations and extensions of the work are described briefly, and full references are provided. Further variations and generalizations may be found in the exercises, which are included at the end of each chapter.

It is a great pleasure to record my gratitude to all those who have helped with this tract in any way. I am particularly indebted to Prof Roy Davies for his careful criticism of the manuscript and for allowing me access to unpublished material, and to Dr Hallard Croft for his detailed suggestions and for help with reading the proofs. I am also most grateful to Prof B.B. Mandelbrot, Prof J.M. Marstrand, Prof P. Mattila and Prof C.A. Rogers for useful comments and discussions.

I should like to thank Mrs Maureen Woodward and Mrs Rhoda Rees for typing the manuscript, and also David Tranah and Sheila Shepherd of Cambridge University Press for seeing the book through its various stages of publication. Finally, I must thank my wife, Isobel, for finding time to read an early draft of the book, as well as for her continuous encouragement and support.

Introduction

The geometric measure theory of sets of integral and fractional dimension has been developed by pure mathematicians from early in this century. Recently there has been a meteoric increase in the importance of fractal sets in the sciences. Mandelbrot (1975, 1977, 1982) pioneered their use to model a wide variety of scientific phenomena from the molecular to the astronomical, for example: the Brownian motion of particles, turbulence in fluids, the growth of plants, geographical coastlines and surfaces, the distribution of galaxies in the universe, and even fluctuations of price on the stock exchange. Sets of fractional dimension also occur in diverse branches of pure mathematics such as the theory of numbers and non-linear differential equations. Many further examples are described in the scientific, philosophical and pictorial essays of Mandelbrot. Thus what originated as a concept in pure mathematics has found many applications in the sciences. These in turn are a fruitful source of further problems for the mathematician. This tract is concerned primarily with the geometric theory of such sets rather than with applications.

The word 'fractal' was derived from the latin *fractus*, meaning broken, by Mandelbrot (1975), who gave a 'tentative definition' of a fractal as a set with its Hausdorff dimension strictly greater than its topological dimension, but he pointed out that the definition is unsatisfactory as it excludes certain highly irregular sets which clearly ought to be thought of in the spirit of fractals. Hitherto mathematicians had referred to such sets in a variety of ways – 'sets of fractional dimension', 'sets of Hausdorff measure', 'sets with a fine structure' or 'irregular sets'. Any rigorous study of these sets must also contain an examination of those sets with equal topological and Hausdorff dimension, if only so that they may be excluded from further discussion. I therefore make no apology for including such regular sets (smooth curves and surfaces, etc.) in this account.

Many ways of estimating the 'size' or 'dimension' of 'thin' or 'highly irregular' sets have been proposed to generalize the idea that points, curves and surfaces have dimensions of 0, 1 and 2 respectively. Hausdorff dimension, defined in terms of Hausdorff measure, has the overriding advantage from the mathematician's point of view that Hausdorff measure *is* a measure (i.e. is additive on countable collections of disjoint sets).

Unfortunately the Hausdorff measure and dimension of even relatively simple sets can be hard to calculate; in particular it is often awkward to obtain lower bounds for these quantities. This has been found to be a considerable drawback in physical applications and has resulted in a number of variations on the definition of Hausdorff dimension being adopted, in some cases inadvertently.

Some of these alternative definitions are surveyed and compared with Hausdorff dimension by Hurewicz & Wallman (1941), Kahane (1976), Mandelbrot (1982, Section 39), and Tricot (1981, 1982). They include entropy, see Hawkes (1974), similarity dimension, see Mandelbrot (1982), and the local dimension and measure of Johnson & Rogers (1982). It would be possible to write a book of this nature based on any such definition, but Hausdorff measure and dimension is, undoubtedly, the most widely investigated and the most widely used.

The idea of defining an outer measure to extend the notion of the length of an interval to more complicated sets of real numbers is surprisingly recent. Borel (1895) used measures to estimate the size of sets to enable him to construct certain pathological functions. These ideas were taken up by Lebesgue (1904) as the underlying concept in the construction of his integral. Carathéodory (1914) introduced the more general 'Carathéodory outer measures'. In particular he defined '1-dimensional' or 'linear' measure in n-dimensional Euclidean space, indicating that s-dimensional measure might be defined similarly for other integers s. Hausdorff (1919) pointed out that Carathéodory's definition was also of value for *non-integral* s. He illustrated this by showing that the famous 'middle-third' set of Cantor had positive, but finite, s-dimensional measure if $s = \log 2/\log 3 = 0.6309\ldots$. Thus the concept of sets of fractional dimension was born, and Hausdorff's name was adopted for the associated dimension and measure.

Since then a tremendous amount has been discovered about Hausdorff measures and the geometry of Hausdorff-measurable sets. An excellent account of the intrinsic measure theory is given in the book by Rogers (1970), and a very general approach to measure geometry may be found in Federer's (1969) scholarly volume, which diverges from us to cover questions of surface area and homological integration theory.

Much of the work on Hausdorff measures and their geometry is due to Besicovitch, whose name will be encountered repeatedly throughout this book. Indeed, for many years, virtually all published work on Hausdorff measures bore his name, much of it involving highly ingenious arguments. More recently his students have made many further major contributions. The obituary notices by Burkill (1971) and Taylor (1975) provide some idea of the scale of Besicovitch's influence on the subject.

It is clear that Besicovitch intended to write a book on geometric measure theory entitled *The Geometry of Sets of Points*, which might well have resembled this volume in many respects. After Besicovitch's death in 1970, Prof Roy Davies, with the assistance of Dr Helen Alderson (who died in 1972), prepared a version of what might have been Besicovitch's 'Chapter 1'. This chapter was not destined to have any sequel, but it has had a considerable influence on the early parts of the present book.

In our first chapter we define Hausdorff measure and investigate its basic properties. We show how to calculate the Hausdorff dimension and measure of sets in certain straightforward cases.

We are particularly interested in sets of dimension s which are *s-sets*, that is, sets of non-zero but finite s-dimensional Hausdorff measure. The geometry of a class of set restricted only by such a weak condition must inevitably consist of a study of the neighbourhood of a general point. Thus the next three chapters discuss local properties: the density of sets at a point, and the directional distribution of a set round each of its points, that is, the question of the existence of tangents. Sets of fractional and integral dimension are treated separately. Sets of fractional dimension are necessarily fractals, but there is a marked contrast between the regular 'curve-like' or 'surface-like' sets and the irregular 'fractal' sets of integral dimension.

Chapter 5 introduces the powerful technique of net measures. This enables us to show that any set of infinite s-dimensional Hausdorff measure contains an s-set, allowing the theory of s-sets to be extended to more general sets as required. Net measures are also used to investigate the Hausdorff measures of Cartesian products of sets.

The next chapter deals with the projection of sets onto lower-dimensional subspaces. Potential-theoretic methods are introduced as an alternative to a direct geometric approach for parts of this work.

Chapter 7 discusses the 'Kakeya problem, of finding sets of smallest measure inside which it is possible to rotate a segment of unit length. A number of variants are discussed, and the subject is related by duality to the projection theorems of the previous chapter, as well as to harmonic analysis.

The final chapter contains a miscellany of examples that illustrate some of the ideas met earlier in the book.

References are listed at the end of the book and are cited by date. Further substantial bibliographies may be found in Federer (1947, 1969), Rogers (1970) and Mandelbrot (1982).

Notation

With the range of topics covered, particularly in the final chapter, it is impossible to be entirely consistent with the use of notation. In general, symbols are defined when they are first introduced; these notes are intended only as a rough guide.

We work entirely in n-dimensional Euclidean space, $\mathbb{R}^n$. Points of $\mathbb{R}^n$, which are sometimes thought of in the vectorial sense, are denoted by small letters, x, y, z etc. Occasionally we write (x, y) for Cartesian coordinates. Capitals, E, F, Γ, etc. are used for subsets of $\mathbb{R}^n$, and script capitals, $\mathscr{C}$, $\mathscr{V}$, $\mathscr{I}$, for families of sets. We use the convention that the set-inclusion symbol $\subset$ allows the possibility of equality. The diameter of the set E is denoted by $|E|$, though, when the sense is clear, the modulus sign also denotes the length of a vector in the usual way, thus $|x - y|$ is the distance between the points x and y. Constants, $b, c, c_1, \varepsilon, \delta$, and indices, i, j, k, are also denoted by lower case letters which may be subscripted.

The following list may serve as a reminder of the notation in more frequent use.

Sets

$\mathbb{R}^n$	n-dimensional Euclidean space.
$B_r(x)$	closed disc or ball, centre x and radius r.
$S_r(x, \theta, \phi)$	sector of angle ϕ and radius r.
$C_r(x, I)$	double sector.
$R(x, y)$	common region of the circle-pair with centres x and y.
$G_{n,k}$	Grossmann manifold of k-dimensional subspaces of $\mathbb{R}^n$.
$L(a, b)$, $L(E)$	line sets.
$\bar{E}$, int E	topological closure, respectively interior, of E.
$[E]_\delta$	the δ-parallel body to E.

Mappings

proj_θ, proj_Π	orthogonal projection onto the line in direction θ, resp. the plane Π.
$\hat{f}, \hat{\mu}$	Fourier transforms of the function f and measure μ.
$f \circ g$	composition of the mappings, g followed by f.

Measures etc.

$\mathscr{H}^s$	s-dimensional Hausdorff measure or outer measure.
$\mathscr{L}^n$	n-dimensional Lebesgue measure.
$\mathscr{M}^s$	s-dimensional comparable net measure.
$\mathscr{H}^s_\delta, \mathscr{M}^s_\delta$	δ-outer measures used in constructing $\mathscr{H}^s$ and $\mathscr{M}^s$.
$\mathscr{L}(\Gamma)$	length of the curve Γ.
$\dim E$	Hausdorff dimension of E.
ϕ_t, C_t, I_t	t-potential, capacity, energy.

Densities

$D^s(E, x)$	density of E at x.
$\underline{D}^s(E, x)$, $\bar{D}^s(E, x)$,	lower, upper densities.
$\bar{D}^s_c(E, x)$	upper convex density.
$\underline{D}^s(E, x, \theta, \phi)$	lower angular density, etc.

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