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Albert N. Shiryaev

ESSENTIALS OF STOCHASTIC FINANCE

Facts, Models, Theory

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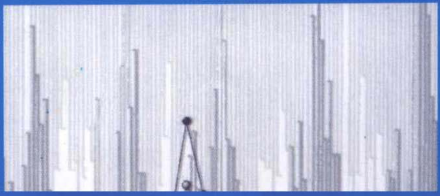

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Facts, Models, Theory

Albert N. Shiryaev
Steklov Mathematical Institute and Moscow State University

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ESSENTIALS OF STOCHASTIC FINANCE

Facts, Models, Theory

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Foreword

The author's intention was:

- *to select and expose subjects that can be necessary or useful to those interested in stochastic calculus and pricing in models of financial markets operating under uncertainty*;
- *to introduce the reader to the main concepts, notions, and results of stochastic financial mathematics*;
- *to develop applications of these results to various kinds of calculations required in financial engineering.*

The author considered it also a major priority to answer the requests of teachers of financial mathematics and engineering by making a bias towards probabilistic and statistical ideas and the methods of stochastic calculus in the analysis of *market risks*.

The subtitle "Facts, Models, Theory" appears to be an adequate reflection of the text structure and the author's style, which is in large measure a result of the 'feedback' with students attending his lectures (in Moscow, Zürich, Aarhus, ...).

For instance, an audience of mathematicians displayed always an interest not only in the mathematical issues of the 'Theory', but also in the 'Facts', the particularities of real financial markets, and the ways in which they operate. This has induced the author to devote the *first chapter* to the description of the key objects and structures present on these markets, to explain there the goals of financial theory and engineering, and to discuss some issues pertaining to the history of probabilistic and statistical ideas in the analysis of financial markets.

On the other hand, an audience acquainted with, say, securities markets and securities trading showed considerable interest in various classes of stochastic processes used (or considered as prospective) for the construction of models of the dynamics of financial indicators (prices, indexes, exchange rates, ...) and important for calculations (of risks, hedging strategies, rational option prices, etc.).

This is what we describe in the *second* and the *third chapters*, devoted to stochastic 'Models' both for discrete and continuous time.

The author believes that the discussion of stochastic processes in these chapters will be useful to a broad range of readers, not only to the ones interested in financial mathematics.

We emphasize here that in the discrete-time case, we usually start in our description of the evolution of stochastic sequences from their *Doob decomposition* into *predictable* and *martingale* components. One often calls this the 'martingale approach'. Regarded from this standpoint, it is only natural that martingale theory can provide financial mathematics and engineering with useful tools.

The concepts of 'predictability' and 'martingality' permeating our entire exposition are incidentally very natural from economic standpoint. For instance, such economic concepts as *investment portfolio* and *hedging* get simple mathematical definitions in terms of 'predictability', while the concepts of *efficiency* and *absence of arbitrage* on a financial market can be expressed in the mathematical language, by making references to martingales and martingale measures (the *First fundamental theorem* of asset pricing theory; Chapter V, § 2b).

Our approach to the description of stochastic sequences on the basis of the Doob decomposition suggests that in the *continuous-time* case one could turn to the (fairly broad) class of *semimartingales* (Chapter III, § 5a). Representable as they are by sums of processes of bounded variation ('slowly changing' components) and local martingales (which can often be 'fast changing', as is a Brownian motion, for example), semimartingales have a remarkable property: one can define stochastic integrals with respect to these processes, which, in turn, opens up new vistas for the application of stochastic calculus to the construction of models in which financial indexes are simulated by such processes.

The *fourth* ('statistical') *chapter* must give the reader a notion of the statistical 'raw material' that one encounters in the empirical analysis of financial data.

Based mostly on currency cross rates (which are established on a global, probably the largest, financial market with daily turnover of several hundred billion dollars) we show that the 'returns' (see (3) in Chapter II, § 1a) have distribution densities with 'heavy tails' and strong 'leptokurtosis' around the mean value. As regards their behavior in time, these values are featured by the 'cluster property' and 'strong aftereffect' (we can say that 'prices keep memory of their past'). We demonstrate the fractal structure of several characteristic of the volatility of the 'returns'.

Of course, one must take all this into account if one undertakes a construction of a model describing the actual dynamics of financial indexes; this is extremely important if one is trying to *foresee* their development in the future.

'Theory' in general and, in particular, *arbitrage theory* are placed in the *fifth chapter* (discrete time) and the *seventh chapter* (continuous time).

Central points there are the *First* and the *Second fundamental asset pricing theorems.*

The *First theorem* states (more or less) that a financial market is *arbitrage-free* if and only if there exists a so-called *martingale* (*risk-neutral*) *probability measure* such that the (discounted) prices make up a martingale with respect to it. The *Second theorem* describes arbitrage-free markets with property of *completeness*, which ensures that one can build an investment portfolio of value replicating *faithfully* any given pay-off.

Both theorems deserve the name *fundamental* for they assign a precise mathematical meaning to the economic notion of an 'arbitrage-free' market on the basis of (well-developed) *martingale theory.*

In the *sixth* and the *eighth chapters* we discuss pricing based on the First and the Second fundamental theorems. Here we follow the tradition in that we pay much attention to the calculation of rational prices and hedging strategies for various kinds of (European or American) *options*, which are *derivative* financial instruments with best developed pricing theory. Options provide a perfect basis for the understanding of the general principles and methods of pricing on arbitrage-free markets.

Of course, the author faced the problem of the choice of 'authoritative' data and the mode of presentation.

The above description of the contents of the eight chapters can give one a measure for gauging the spectrum of selected material. However, for all its bulkiness, our book leaves aside many aspects of financial theory and its applications (e.g., the classical theories of von Neumann–Morgenstern and Arrow–Debreu and their updated versions considering investors' behavior delivering the maximum of the 'utility function', and also computational issues that are important for applications).

As the reader will see, the author often takes a lecturer's stance by making comments of the 'what-where-when' kind. For discrete time we provide the proofs of essentially all main results. On the other hand, in the continuous-time case we often content ourselves with the statements of results (of martingale theory, stochastic calculus, etc.) and refer to a suitable source where the reader can find the proofs.

The suggestion that the author could write a book on financial mathematics for *World Scientific* was put forward by Prof. Ole Barndorff-Nielsen at the beginning of 1995. Although having accepted it, it was not before summer that the author could start drafting the text. At first, he had in mind to discuss only the *discrete-time* case. However, as the work was moving on, the author was gradually coming to the belief that he could not give the reader a full picture of financial mathematics and engineering without touching upon the *continuous-time* case. As a result, we discuss both cases, discrete and continuous.

This book consists of two parts. The first ('Facts. Models') contains Chapters I–IV. The second ('Theory') includes Chapters V–VIII.

The writing process took around two years. Several months went into typesetting, editing, and preparing a camera-ready copy. This job was done by I. L. Legostaeva, T. B. Tolozova, and A. D. Izaak on the basis of the Information and Publishing Sector of the Department of Mathematics of the Russian Academy of Sciences. The author is particularly indebted to them all for their expertise and selfless support as well as for the patience and tolerance they demonstrated each time the author came to them with yet another 'final' version, making changes in the already typeset and edited text.

The author acknowledges the help of his friends and colleagues, in Russia and abroad; he is also grateful to the Actuarial and Financial Center in Moscow, VW-Stiftung in Germany, the Mathematical Research Center and the Center for Analytic Finance in Aarhus (Denmark), INTAS, and the A. Lyapunov Institute in Paris and Moscow for their support and hospitality.

Moscow
1995 – 1997

A. Shiryaev
Steklov Mathematical Institute
Russian Academy of Sciences
and
Moscow State University

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