MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

Submanifolds and Holonomy

Second Edition

Jürgen Berndt
Sergio Console
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Für meine Eltern Marianne und Erhard
Ai miei genitori Franca e Ettore
A mis padres Rosa y Enrique

Preface to the First Edition

The geometry of curves and surfaces has attracted mathematicians, physicists, and other scientists for many centuries. Starting from simple geometric observations, mathematicians produce highly sophisticated theories that often lead not just to a deeper understanding of the observations made at the beginning, but also to further questions. Curves are one-dimensional objects and surfaces have two dimensions. One question that often arises is: What happens in higher dimensions? This is a natural question, since experience tells us that, in many instances, more than two dimensions are relevant. The generalizations of curves and surfaces to higher dimensions are submanifolds.

In this book we deal with particular questions about the geometry of submanifolds.

For Jürgen Berndt, the gateway to this area has been the classification by Élie Cartan of isoparametric hypersurfaces in real hyperbolic spaces. In his doctorate thesis he investigated the analogous problem for complex hyperbolic spaces. Surprisingly, a full classification is still not known, and recent results show that this problem is much more difficult than expected. These recent results stem from the author's investigations about isometric actions with an orbit of codimension one, so-called cohomogeneity one actions. Cohomogeneity one actions are currently of interest in Riemannian geometry for the construction of metrics with special properties, for instance, Einstein metrics, metrics with special holonomy, and metrics of positive sectional curvature. The investigation of actions on manifolds and the geometry of their orbits is a central theme in his research.

Submanifold geometry is the primary research topic of Sergio Console. He has been particularly interested in the interaction of algebraic and geometric methods for the study of the Riemannian geometry and the topology of submanifolds of space forms with simple geometric invariants, for example, isoparametric or homogeneous submanifolds. In particular, he learned from the third author how to use holonomy methods in submanifold geometry, a theme he discussed much with the first author in 1995 when they both worked at the University of Cologne. This was the beginning of the plan to write the present monograph, and collaboration on this project started when all the authors met in Turin in 1997.

Carlos Olmos is mainly interested in local and global submanifold geometry in space forms, in particular in problems related to the so-called normal holonomy that combines local and global methods. He is also interested in Riemannian and Lorentzian geometry. The subjects of his doctoral thesis, directed by Cristian Sánchez, motivated most of his research.

Many available textbooks deal with the geometry of curves and surfaces, the classical topic for introductory courses to differential geometry at universities. In contrast, only a few books deal with submanifolds of higher dimensions. Although many books on differential geometry contain chapters about submanifolds, these chapters are often quite short and contain only basic material. A standard reference for submanifold geometry has been *Geometry of Submanifolds* by Bang-yen Chen, but this book was written in 1973 and concerned research problems that were of interest at that time. Books dealing with more recent problems from submanifold geometry are *Critical Point Theory and Submanifold Geometry* (1988) by Richard Palais and Chuu-lian Terng, *Submanifolds and Isometric Immersions* (1990) by Marcos Dajczer et al., *Tubes* (1990) by Alfred Gray, and *Lie Sphere Geometry with Applications to Submanifolds* (1992) by Thomas Cecil. To some extent, these books deal with topics that also appear in our book, but, for these problems, our approach is different and relies on methods involving the holonomy group of the normal connection of a submanifold. These methods originated from the Normal Holonomy Theorem that was proved by the third author in 1990. The Normal Holonomy Theorem is the analogue for submanifold geometry in space forms of Marcel Berger's classification of holonomy groups of Riemannian connections on manifolds. Since 1990, normal holonomy has developed as a powerful tool in submanifold geometry. The purpose of this book is to present a modern and self-contained introduction to submanifold geometry with emphasis on topics where the tool of normal holonomy had great impact. This book is aimed at researchers and graduate students in mathematics, in particular in geometry, and could be used as a textbook for an advanced graduate course.

We briefly describe the contents of this book. Until now, the main applications of normal holonomy concern submanifolds of space forms, that is, manifolds of constant sectional curvature. For this reason, we first present an introduction to submanifolds in space forms and discuss in detail the fundamental results about such submanifolds. Important examples of submanifolds of Euclidean spaces are orbits of linear Lie group actions, and, for this reason, we investigate in great detail the geometry of such orbits. Then we introduce the concept of normal holonomy and present the Normal Holonomy Theorem together with its proof and some applications. In great detail, we apply the tool of normal holonomy to study isoparametric submanifolds and their focal manifolds, orbits of linear Lie group actions and homogeneous submanifolds, and homogeneous structures on submanifolds. At the end of the book we discuss generalizations to submanifolds of Riemannian manifolds, in particular of Riemannian symmetric spaces. In an appendix, we summarize the necessary facts about Riemannian manifolds, Lie groups and Lie algebras, homogeneous spaces, symmetric spaces, and flag manifolds, which the reader might find helpful.

Several proofs presented in the book have never appeared in the literature. For instance, we present a new proof of Cartan's theorem about the existence of totally geodesic submanifolds of Riemannian manifolds, a result that is hard to find in the literature. An advantage of this book is that it contains much material that is currently accessible only in a large number of published articles in various journals. The book also contains a number of open problems that might attract the reader.

Of course, there are many interesting and fascinating problems in submanifold geometry that are not touched on in this book. The reason is simply that there are too many of these problems. Our selection of topics for this book has been motivated by normal holonomy and, naturally, also by personal taste and interest.

To produce most of the illustrations we used the software SUPERFICIES by Angel Montesinos Amilibia of Universidad de Valencia. SUPERFICIES is freely distributed, with source code, under GNU General Public License and is available at ftp://topologia.geomet.uv.es/pub/montesin/.

We would like to thank Simon Chiossi, Antonio Di Scala, Anna Fino, Sergio Garbiero, and Simon Salamon for their careful reading of parts of the manuscript and for their suggestions for improvements.

Preface to the Second Edition

The second edition contains five new chapters

7 Normal holonomy of complex submanifolds
8 The Berger-Simons holonomy theorem
9 The skew-torsion holonomy system
12 Polar actions on symmetric spaces of compact type
13 Polar actions on symmetric spaces of noncompact type

and three new sections or subsections

2.2 Existence of slices and principal orbits for isometric actions
10.3.1 Maximal totally geodesic submanifolds
11.1.6 The index of symmetric spaces

The contents of Section 9.5 in the first edition has been incorporated into the new Chapters 12 and 13.

The second edition contains many updates on developments since the first edition was published in 2003. We also corrected some misprints and errors in the first edition.

The authors would like to thank the following colleagues for pointing out errors in the first edition and for suggesting improvements: Christian Boltner, Antonio J. Di Scala, Claudio Gorodski, Ernst Heintze, Jong Ryul Kim, Andreas Kollross, Guilherme Machado de Freitas, Bogdan Popescu, Gudlaugur Thorbergsson, Francisco Vittone, and Kerstin Weinl.

Sergio Console (1965–2013)

To our deep sorrow, our friend, colleague, and co-author Sergio Console passed away unexpectedly on November 4, 2013. It was a great pleasure for us to work with Sergio over many years and to be together with him on many occasions to discuss mathematics and enjoy many other aspects of life.

Jürgen Berndt and Carlos Olmos

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