Bombay Lectures on

HIGHEST WEIGHT REPRESENTATIONS *of* INFINITE DIMENSIONAL LIE ALGEBRAS

Victor G. Kac, Ashok K. Raina & Natasha Rozhkovskaya

Second Edition

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PREFACE

This book is a write-up of a series of lectures given by the first author at the Tata Institute, Bombay, during December 1985–January 1986.

The dominant theme of these lectures is the idea of a highest weight representation. This idea goes through four different incarnations.

The first is the canonical commutation relations of the infinite-dimensional Heisenberg algebra (= oscillator algebra). Although this example is extremely simple, it not only contains the germs of the main features of the theory, but also serves as a basis for most of the constructions of representations of infinite-dimensional Lie algebras.

The second is the highest weight representations of the Lie algebra $g\ell_\infty$ of infinite matrices, along with their applications to the theory of soliton equations, discovered by Sato and Date-Jimbo-Kashiwara-Miwa. Here the main point is the isomorphism between the vertex and the "Dirac sea" realizations of the fundamental representations of $g\ell_\infty$, a kind of a Bose-Fermi correspondence.

The third is the unitary highest weight representations of the affine Kac-Moody (= current) algebras. Since there is now a book devoted to the theory of Kac-Moody algebras, it was decided to devote to them a minimum attention. In the lectures affine algebras play a prominent role only in the Sugawara construction as the main tool in the study of the fourth incarnation of the main idea, the theory of highest weight representations of the Virasoro algebra.

The main results of the representation theory of the Virasoro algebra which are proved in these lectures are the Kac determinant formula and the unitarity of the "discrete series" representations of Belavin-Polyakov-Zamolodchikov and Friedan-Qiu-Shenker.

We hope that this elementary introduction to the subject, written by a mathematician and a physicist, will prove useful to both mathematicians and physicists. To mathematicians, since it illustrates, on important examples, the interaction of the key ideas of the representation theory of infinite-dimensional Lie algebras; and to physicists, since this theory is turning before our very eyes into an important component of such domains of theoretical physics as soliton theory, theory of two-dimensional statistical models, and string theory.

Throughout the book, the base field is the field of complex numbers $\mathbb{C}$, unless otherwise stated, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ the set of integers, and $\mathbb{Z}_+$ (resp. $\mathbb{N}$) the set of nonnegative (resp. positive) integers.

The authors wish to thank the participants of the lectures, especially S. M. Roy and S. R. Wadia, for valuable suggestions and comments.

PREFACE TO THE SECOND EDITION

The first edition of this book was based on the series of lectures, given by the first author at the Tata Institute of Fundamental Research, Mumbai (Bombay), in the winter of 1985–1986, just before the birth of the theory of vertex algebras.

As a consequence of the development of this theory, many results and constructions of representation theory of infinite-dimensional Lie algebras have been greatly extended and clarified. Therefore, when the first author was again invited to visit the TIFR some eighteen years later, it was quite natural to lecture on vertex algebras.

The second edition of the book consists of two parts. The first part (Lectures 1-12) contains the largely unchanged text of the first edition, while the second part (Lectures 13-18) is an extended write-up of the lectures, delivered at the TIFR in January 2003.

The basic idea of these lectures was to demonstrate how the key notions of the theory of vertex algebras - such as the quantum fields, their normal ordered product and λ-bracket, energy-momentum field and conformal weight, untwisted and twisted representations - simplify and clarify the constructions of the first edition of the book.

The first author wishes to thank M. Shubin for a correction to Proposition 1.2, and B. Bakalov for a discussion on his approach to twisted representations, used in the book.

CONTENTS

LECTURE 1

In this series of lectures we shall demonstrate some basic concepts and methods of the representation theory of infinite-dimensional Lie algebras on four main examples:

1. The oscillator algebra
2. The Virasoro algebra
3. The Lie algebra $g\ell_\infty$
4. The affine Kac-Moody algebras.

In fact, the interplay between these examples is one of the key methods of the theory.

1.1. The Lie algebra $\mathfrak{d}$ of complex vector fields on the circle

We shall first consider the Virasoro algebra, which is playing an increasingly important role in theoretical physics. It is also a natural algebra to consider from a mathematical point of view as it is a central extension of the complexification of the Lie algebra *Vect* S^1 of (real) vector fields on the circle S^1. We shall start by finding the structure of *Vect* S^1 and later consider its central extensions.

Any element of *Vect* S^1 is of the form $f(\theta)\, d/d\theta$, where $f(\theta)$ is a smooth real-valued function on S^1, with θ a parameter on S^1 and $f(\theta+2\pi) = f(\theta)$. The Lie bracket of vector fields is:

$$\left[f(\theta)\frac{d}{d\theta},\, g(\theta)\frac{d}{d\theta}\right] = (fg' - f'g)(\theta)\frac{d}{d\theta}\,,$$

where prime stands for the derivative. A basis (over $\mathbb{R}$) for *Vect* S^1 is provided by the vector fields

$$\frac{d}{d\theta}, \quad \cos(n\theta)\frac{d}{d\theta}, \quad \sin(n\theta)\frac{d}{d\theta} \quad (n = 1, 2, \ldots).$$

To avoid convergence questions we consider this as a vector space basis, so that $f(\theta)$, $g(\theta)$ are arbitrary trigonometric polynomials, and take its linear span over $\mathbb{C}$ as this permits us to introduce $\exp(in\theta)$ instead of $\cos(n\theta)$ and $\sin(n\theta)$. We thus obtain a complex Lie algebra, denoted by $\mathfrak{d}$, with a basis

$$d_n = i\exp(in\theta)\frac{d}{d\theta} = -z^{n+1}\frac{d}{dz} \quad (n \in \mathbb{Z}) \tag{1.1}$$

where $z = \exp(i\theta)$. These elements satisfy the following commutation relations:

$$[d_m, d_n] = (m - n)d_{m+n} \quad (m, n \in \mathbb{Z}). \tag{1.2}$$

The Lie algebra *Vect* S^1 can be considered as the Lie algebra of the group G of orientation preserving diffeomorphisms of S^1. If ζ_1, ζ_2 are two elements of G then their product is defined by composition:

$$(\zeta_1 \cdot \zeta_2)(z) = \zeta_1(\zeta_2(z))$$

for each $z = \exp(i\theta)$ on S^1. If $f(z)$ is an element of the vector space of smooth complex-valued functions on S_1, then $\gamma \in G$ acts on $f(z)$ by

$$\pi(\gamma)f(z) = f(\gamma^{-1}(z)). \tag{1.3}$$

This clearly defines a representation of G. We take γ close to the identity (as physicists do):

$$\gamma(z) = z(1 + \epsilon(z)) = z + \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1} \tag{1.4a}$$

where we have made a Laurent (or Fourier) expansion of $\epsilon(z)$ and the ϵ_n are to be retained up to first order only. Then

$$\gamma^{-1}(z) = z - \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1} \tag{1.4b}$$

and

$$\pi(\gamma)f(z) = f\left(z - \sum_n \epsilon_n z^{n+1}\right) = \left(1 + \sum_n \epsilon_n d_n\right) f(z) \tag{1.5}$$

where the d_n are defined by (1.1). This shows that the d_n form a (topological) basis of the complexification of the Lie algebra of G.

In the following we shall consider the complex Lie algebra $\mathfrak{d}$ and view $\mathfrak{d} \cap \mathit{Vect}\, S^1$ as the subalgebra (over $\mathbb{R}$) of real elements. One way to do this is to regard $\mathfrak{d} \cap \mathit{Vect}\, S^1$ as the subalgebra of fixed points for the operation of complex conjugation under which d_n maps to $-d_{-n}$ and a scalar λ to its complex conjugate $\overline{\lambda}$. It is more convenient, however, to introduce a slightly different operation defined by:

$$\omega(d_n) = d_{-n}\,, \tag{1.6a}$$

$$\omega(\lambda x) = \overline{\lambda}\omega(x)\,, \tag{1.6b}$$

so that

$$\omega([x, y]) = [\omega(y), \omega(x)] \tag{1.6c}$$

where $x, y \in \mathfrak{d}$, $\lambda \in \mathbb{C}$. Thus ω is an antilinear anti-involution having the algebraic properties of Hermitian conjugation. Now $\mathfrak{d} \cap \mathit{Vect}\, S^1$ consists of elements of $\mathfrak{d}$ fixed under $-\omega$.

The purely algebraic operation ω on $\mathfrak{d}$ can become an adjoint operation with respect to a suitable scalar product if we have a representation of $\mathfrak{d}$ in some vector space. Suppose that we have a unitary representation of the group G on a vector space V with a positive-definite Hermitian form $\langle \cdot \,|\, \cdot \rangle$. Identifying elements of G with corresponding operators, we have:

$$\langle g(u) \,|\, g(v) \rangle = \langle u \,|\, v \rangle \quad \text{for} \quad g \in G\,,\ u, v \in V\,.$$

Going over to the Lie algebra, this means that

$$\langle x(u) \,|\, v \rangle = -\langle u \,|\, x(v) \rangle \quad \text{for} \quad x \in \mathit{Vect}\, S^1\,,$$

and for any $x \in \mathfrak{d}$:

$$\langle x(u) \,|\, v \rangle = \langle u \,|\, \omega(x)(v) \rangle\,. \tag{1.7}$$

This motivates the following definitions:

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra and let ω be an antilinear anti-involution on $\mathfrak{g}$, i.e. an $\mathbb{R}$-linear involution satisfying (1.6b and c). Let V be a representation space of $\mathfrak{g}$ and $\langle \cdot \,|\, \cdot \rangle$ an Hermitian form on V. We say that $\langle\cdot|\cdot\rangle$ is *contravariant* if (1.7) holds for all $x \in \mathfrak{g}$, and $u, v \in V$. When $\langle\cdot|\cdot\rangle$ is nondegenerate, this means that

$$x^\dagger = \omega(x) \quad \text{for all} \quad x \in \mathfrak{g}\,. \tag{1.7'}$$

Here and further $x^\dagger$ stands for the Hermitian adjoint of the operator x. We further say that this representation is *unitary* if in addition

$$\langle v\,|\,v\rangle > 0 \quad \text{for all} \quad v \in V\,, \quad v \neq 0\,.$$

1.2. Representations $V_{\alpha,\beta}$ of $\mathfrak{d}$

We shall find representations of $\mathfrak{d}$ by considering a suitable vector space on which the group G acts and determining the action of G in this space for elements close to the identity. Using (1.5) we shall determine the action of d_n in this vector space.

Let $V_{\alpha,\beta}$ denote that space of 'densities' of the form $P(z)z^\alpha(dz)^\beta$, where α and β are complex numbers and $P(z)$ is an arbitrary polynomial in z and z^{-1}. A basis for $V_{\alpha,\beta}$ is given by the set of vectors

$$v_k = z^{k+\alpha}(dz)^\beta \quad (k \in \mathbb{Z})\,. \tag{1.8}$$

From (1.3),

$$\pi(\gamma)v_k = (\gamma^{-1}(z))^{k+\alpha}(d\gamma^{-1}(z))^\beta$$

and if γ is of the form (1.4a), we can use (1.4b) for $\gamma^{-1}(z)$. Thus

$$\begin{aligned}\pi(\gamma)v_k &= \left(z - \sum_n \epsilon_n z^{n+1}\right)^{k+\alpha} \left(\left(1 - \sum_n \epsilon_n (n+1) z^n\right) dz\right)^\beta \\ &= \left(1 - (k+\alpha)\sum_n \epsilon_n z^n\right)\left(1 - \beta\sum_n \epsilon_n (n+1) z^n\right) z^{k+\alpha}(dz)^\beta \\ &= \left(1 - \sum_n \epsilon_n (k+\alpha+\beta n+\beta) z^n\right) z^{k+\alpha}(dz)^\beta\,.\end{aligned}$$

Comparing with (1.5) we see that

$$d_n(v_k) = -(k+\alpha+\beta+\beta n)v_{n+k} \quad (n, k \in \mathbb{Z})\,. \tag{1.9}$$