Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1066

Numerical Analysis

Proceedings, Dundee 1983

Edited by David F. Griffiths

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Proceedings of the 10th Biennial Conference
held at Dundee, Scotland, June 28 – July 1, 1983

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1. Numerical analysis — Congresses

Editor

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Preface

The Tenth Dundee Biennial Conference on Numerical Analysis, held at the University of Dundee, Scotland on the four days 28 June - 1 July, 1983, attracted over 200 participants from 25 countries. The organizers were again fortunate in gaining the services of 16 eminent numerical analysts covering a broad spectrum of the subject and it is their papers which appear in these notes. Unfortunately Professor Dupont's contribution was not available at the time of going to press. In addition to the invited talks, short contributions were solicited and 69 of these were presented at the conference in three parallel sessions. A complete list of these submitted papers, together with authors' addresses, is also given here.

I would like to take this opportunity of thanking Professor Dr L Collatz who, as after dinner speaker, kept the audience greatly amused with anecdotes, some true, some with only a grain of truth and others apocryphal, concerning many well-known mathematicians. It is also a pleasure to thank all the speakers, the session chairmen and the members of the Mathematical Sciences Department of this University for their contributions and assistance with the successful outcome of this conference. I am particularly indebted to Mrs Dorothy Hargreaves for attending to the considerable task of typing the various documents associated with the conference and for coping so admirably with many of the organizational details.

Financial support for this conference was obtained from the European Research Office of the United States Army. This support is gratefully acknowledged.

Dundee, January 1984.

D F Griffiths

INVITED SPEAKERS

R H Bartels: Department of Computer Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.

A Bjorck: Department of Mathematics, Linkoping University, S-581 83 Linkoping, Sweden.

C de Boor: Mathematics Research Center, University of Wisconsin-Madison, Madison, Wisconsin 53706, USA.

H Brunner: Institut de Mathematiques, University of Fribourg, CH 1700 Fribourg, Switzerland.

T Dupont: Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, Illinois 60637, USA.

D M Gay: Bell Laboratories, 600 Mountain Ave., Murray Hill, NJ 07974, USA.

P W Hemker: Mathematical Centre, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands.

M J D Powell: Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, England.

P A Raviart: Analyse Numérique, Université Pierre et Marie Curie (Paris VI), 4 Place Jussieu, 75230 Paris, Cedex 05, France.

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Ph L Toint: Department of Mathematics, Facultes Universitaires de Namur, Rempart de la Vierge 8, B-5000 Namur, Belgium.

G A Watson: Department of Mathematical Sciences, University of Dundee, Dundee DD1 4HN, Scotland.

M F Wheeler: Mathematical Sciences Department, Rice University, Houston, Texas 77001, USA.

J R Whiteman: Institute for Numerical Mathematics, Brunel University, Kingston Lane, Uxbridge, Middlesex UB8 3PM, England.

SUBMITTED PAPERS

C A Addison and P M Hanson, Department of Computer Science, University of Victoria, P O Box 1700, Victoriam B.C., Canada V8W 2Y2.
An investigation into the stability properties of second derivative methods with perfect square iteration matrices.

J M Aitchison, Computing Laboratory, University of Oxford, 19 Parks Road, Oxford OX1 3PL, England.
The calculation of free surface flows using a conformal transformation and boundary element techniques.

S Amini and D T Wilton, Department of Mathematics, Statistics and Computing, Plymouth Polytechnic, Drake Circus, Plymouth, Devon PL4 8AA, England.
A comparison between boundary element methods for the acoustic radiation problem.

H Arndt, Institut fur Angewandte Mathematik, Wegelerstr. 6, D-5300 Bonn, W Germany.
On step size control for Volterra integral equations.

U Ascher, Department of Mathematics, University of British Columbia, Vancouver, Canada.
Collocation methods for singular perturbation problems.

M L Baart, NRIMS-CSIR, P O Box 395, Pretoria 0001, South Africa.
Recursive calculation of curved finite element stiffness matrices.

K Barrett, Mathematics Department, Coventry Polytechnic, Priory Street, Coventry CV1 5FB, England.
A multigrid finite element method for general regions.

L Brutman, Department of Mathematics and Computer Science, University of Haifa, Mount Carmel, Haifa 31999, Israel.
Generalized alternating polynomials, some properties and numerical applications.

P Camino and J M Sanz-Serna, Departmento de Ecuaciones Funcionales, Universidad de Valladolid, Facultad de Ciencias, Valladolid, Spain.
ODE solvers for trajectory problems and applications to PDEs.

C Carter, Department of Mathematics, Trent University, Peterborough, Ontario, Canada K9J 7B8.
Spectral solution of stiff piecewise-linear differential equations occurring in heat transfer through a network.

M M Cecchi, Instituto di Elaborazione del Informasione, Via S Maria 46, 56100 Pisa, Italy.
Spline approximation for linear two point boundary value problems.

A R Conn and N I M Gould, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.
The calculation of directions of negative curvature for non-linear minimization problems.

G J Cooper, School of Mathematics and Physical Sciences, University of Sussex, Brighton BN1 9QH, England.
Algebraic stability for explicit Runge-Kutta methods.

I C Demetriou, DAMTP, University of Cambridge, Silver Street, Cambridge, England.
Piecewise monotonic least squares data fitting.

D B Duncan, Chalk River Nuclear Laboratories, Atomic Energy of Canada, Chalk River, Ontario, Canada K0J 1J0.
Numerical scheme for a groundwater flow problem with radionuclide seepage.

C M Elliott, Mathematics Department, Imperial College, Queens Gate, London SW7 2BX, England.
Ordinary differential inclusions and equations with discontinuous right hand sides.

R England, IIMAS, Universidad Nacional Autónoma México, Apdo Postal 20-726, Delegacion de Alvaro Obregón, 01000 Mexico, D.F.
Implicit multistage methods for ordinary differential equations and their efficient implementation.

Th Fawzy* and I Refat+, *Mathematics Department, Suez Canal University, Ismailia, Egypt, +P O Box 8878, Salmiya, Kuwait.
On the Lacunary interpolation with splines.

Th Fawzy* and I Refat+, *Mathematics Department, Suez Canal University, Ismailia, Egypt, +P O Box 8878, Salmiya, Kuwait.
On the approximate solution of the D E $y'' = f(x,y,y')$.

R Fletcher, Department of Mathematical Sciences, University of Dundee, Dundee DD1 4HN, Scotland.
Expected conditioning.

R Fletcher and S P J Matthews, Department of Mathematical Sciences, University of Dundee, Dundee DD1 4HN, Scotland.
Stable modification of explicit LU factors for simplex updates.

M Friedman and G Erez, Department of Physics, Nuclear Research Center-Negev, Beer-Sheva, P O Box 9001, Israel.
An improved Newtons' method.

C W Gear and O Osterby, Computer Science Department, University of Aarhus, Ny Munkegate, DK8000 Aarhus C, Denmark.
Methods for ordinary differential equations with discontinuities.

R Ge, DAMTP, University of Cambridge, Silver Street, Cambridge CB3 9EW, England.
A filled function method for finding the global minimizer.

H Gerritsen, Delft Hydraulics Laboratory, P O Box 177, 2600 MH Delft, The Netherlands.
Accurate boundary treatment in shallow water flow computations.

J K Gibson, Mathematics Department, Birkbeck College, Malet Street, London WC1, England.
The double secant method for real polynomial equations.

I Gladwell* and C A Addison+, *Mathematics Department, University of Manchester, Manchester M13 9PL, England, +Department of Computer Science, University of Victoria, Victoria, B.C., Canada V8W 2Y2.
Implementation of second derivative methods for implicit second order systems.

K Hebeker, Fachbereich 17 Mathematik/Informatik, Universtat GHS Paderborn, Warburger Str 100, D-4790 Paderborn, W Germany.
A boundary element method for computing 3-D viscous incompressible fluid flows.

G W Hedstrom, Lawrence Livermore Laboratory, University of California, P O Box 808, Livermore, California 94450, USA.
A moving-grid method for tracking shocks.

B M Herbst, Department of Mathematical Sciences, University of Dundee, Dundee DD1 4HN. Scotland.
Stability and the nonlinear Schrodinger equation.

N J Higham, Half Edge Lane, Eccles, Manchester M30 9BA, England.
Upper bounds for the condition number of a triangular matrix.

N Houbak, Numerical Institute, Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark.
SIL - a simulation language.

P J van der Houwen and B P Sommeijer, Mathematical Centre, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands.
Linear multistep methods with minimized truncation error for periodic initial value problems.

W H Hundsdorfer, Department of Mathematics, University of Leiden, Wassenaarseweg 80, Postbus 951, 2300 Ra Leiden, The Netherlands.
B-Stability for semi-implicit methods.

A Iserles, DAMTP, University of Cambridge, Silver Street, Cambridge CB3 9EW, England.
Order stars and finite differences for parabolic PDE's.

K Jonasson, Department of Mathematical Sciences, University of Dundee, Dundee DD1 4HN.
Numerical solution of continuous Chebyshev approximation problems.

P E Koch, Institute of Information, University of Oslo, P O Box 1080, Blindern, Oslo 3, Norway.
A collocation method for singularly perturbed two-point boundary value problems with a turning point.

D P Laurie, NRIMS-CSIR, P O Box 395, Pretoria 0001, South Africa.
Practical error estimates in numerical integration.

P E Manneback, Division of Information and Computing, National Physical Laboratory, Teddington, Middlesex, TW11 OLW, England.
Solution of weighted linear least squares problems with simple structure.

V S Manoranjan, Department of Mathematical Sciences, University of Dundee, Dundee DD1 4HN, Scotland.
Bifurcation studies in reaction-diffusion.

J C Mason, Mathematics Branch, Royal Military College of Science, Shrivenham, Swindon, England.
L_p approximation by real and complex Chebyshev series.

E J W Ter Maten and G L G Sleijpen, Department of Mathematics, University of Utrecht, Budapestlaan 6, P O Box 80.010, 3508 TA Utrecht, The Netherlands.
Stability analysis of high accuracy methods for fourth order parabolic equations.

G Moore, School of Mathematical Sciences, National Institute for Higher Education, Dublin 9, Republic of Ireland.
Divided differences and defect correction for the finite element method.

F D Van Niekerk, Department of Mathematics, University of Pretoria, Pretoria, South Africa.
A global-local finite element method in space-time for a convection-diffusion problem.

J Nocedal and M Overton, Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA.
Numerical methods for solving inverse eigenvalue problems.

F A De Oliveira, Departmento de Matematica, Universidade de Coimbra, 3000-Coimbra, Portugal.
Two-point-boundary-value problems with interval analysis.

G Opfer, Institut fur Angewandte Mathematik, University of Hamburg, Bundesstr. 55 D 2000 Hamburg 13, Germany.
Richardson's iteration for nonsymmetric matrices.

F Patricio, Departmento de Matematica, Universidade de Coimbra, 3000-Coimbra, Portugal.
A class of methods for stiff ordinary differential equations.

P W Pedersen, Department of Mathematics, Technical University of Denmark, DK 2800 Lyngby, Denmark.
Computing square roots with 15 decimal accuracy using only 4 multiplications - and some algorithms.

H J J te Riele and P Schroevers, Mathematical Centre, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands.
A comparison of numerical methods for the linear generalized Abel integral equation.

R A Sack and R Brown, Department of Mathematics, University of Salford, Salford M5 4WT, England.
Vector analogues of Aitken's δ^2-process.

L L Schumaker, Department of Mathematics, University of Texas at Austin, Austin, Texas 78712, USA.
Finding the zeros of splines.

B W Scotney, Department of Mathematics, University of Reading, Whiteknights, Reading RG6 2AX, England.
Optimal error estimates for Petrov-Galerkin methods.

A Sharma, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada.
An extension of a theorem of J L Walsh.

S Sigurdsson, Faculty of Engineering and Science, University of Iceland, Hjardarhagi 2-6, Reykjavik, Iceland.
Construction of generalized Adams methods for stiff ODEs.

G L G Sleijpen and E J W Ter Maten, Department of Mathematics, University of Utrecht, Budapestlaan 6, P O Box 80.010, 3508 TA Utrecht, The Netherlands.
Stability analysis of hopscotch methods for fourth order parabolic equations.

D M Sloan, Department of Mathematics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH., Scotland.
Stability of a boundary condition for a fourth order hyperbolic difference scheme.

M N Spijker, Subfacultad der Wiskunde, University of Leiden, Wassenaarseweg 80, Postbus 9512, 2300 Leiden, The Netherlands.
The existence of solutions to the algebraic equations in implicit Runge-Kutta methods.

J Steinberg, Technion, Israel Institute of Technology, Haifa, Israel.
Computation of stress intensity factors.

R Thatcher, Department of Mathematics, UMIST, P O Box 88, Sackville Street, Manchester M60 1QD., England.
Variational methods for solving free boundary problems.

P G Thomsen and P Norsett, Institute for Numerical Analysis, Technical University of Denmark, DK-2800 Lyngby, Denmark.
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E F Toro and M J O'Carroll, Department of Applied Mathematical Studies, University of Leeds, Leeds LS2 9JT., England.
A Kantarovic computational method for free surface gravity flows.

M van Veldhuizen, Wiskundig Seminarium, Vrije Universiteit, De Boelelaan 1081, Postbus 7161, 1007 MC Amsterdam, The Netherlands.
D-stability and Kaps-Rentrop methods.

A J Wathen and M J Baines, Department of Mathematics, University of Reading, Whiteknights, Reading, Berks., England.
The structure of the moving finite element.

B Werner and A Spence, Department of Mathematics, University of Hamburg, Bundesstr. 55, D-2000 Hamburg 13, W Germany.
Computation of symmetry-breaking bifurcation points.

R A Williamson, DAMTP, University of Cambridge, Silver Street, Cambridge CB3 9EW.
The stability and accuracy of semi-discretised schemes in the numerical solution of hyperbolic differential equations.

T J Ypma, Department of Applied Mathematics, University of Witwatersrand, Johannesburg 2001, S. Africa.
Local convergence in inexact Newton methods.

Y Yuan, DAMTP, University of Cambridge, Silver Street, Cambridge CB3 9EW, England.
An example of only linear convergence of trust region algorithms for nonsmooth optimization.

Z Zlatev, R Berkowicz and L P Prahm, Air Pollution Laboratory, Riso National Laboratory, DK-4000 Roskilde, Denmark.
Selfadaptive time-integration in a code for numerical treatment of an air pollution model.

CONTENTS

Splines in Interactive Computer Graphics

Richard H. Bartels

ABSTRACT

Computer graphics, particularly interactive computer graphics, is not, as the name might imply, concerned with drawing graphs, but rather with the broadest issues of manipulating, transforming, and displaying information in visual format. It is interactive in so far as operations can be carried out in real time – which requires algorithms of high computational efficiency and low complexity.

Splines are a valuable tool in graphics, but they are often applied in a way not used by the mathematician. This difference raises computational issues which the numerical analyst might otherwise never see. This talk will provide a brief introduction to such issues and follow with a study of two current developments.

We begin with a review of the graphics environment, mentioning the modelling and display process and pointing out some of the costly issues. The novel use of splines in interactive graphics comes through the construction of surfaces as weighted averages of selected points, called "control vertices" in which B-splines are taken as the weighting functions. Some examples will illustrate the characteristics of this use of B-splines.

With this background we consider two recent developments. The first is the control-vertex recurrence of Riesenfeld, Cohen, and Lyche; the second is Barsky's work on geometric vs. mathematical continuity, and his introduction of Beta-splines. We will close with some results on current research concerned with a synthesis of these two developments.

1. Introduction

The Computer Graphics Laboratory at the University of Waterloo has embarked on a programme to investigate techniques of potential use for the next generation of computer-aided design systems. The terms of reference are *interactive* and *surface modeling*. The context is not that of fitting curves or surfaces to data or objects which exist – plotting or approximation is not of interest. The context is that of (1) providing a mathematically naive industrial draftsman, sitting before a screen and control panel, with an easy way of creating the mathematical description of an object which does not yet exist, (2) displaying and manipulating images of that object on the screen, (3) modifying the object, and (4) ultimately generating machine-tool commands which will provide a means for producing the object. In design systems of this type splines have been very important in the past, and they are undergoing interesting developments for the future.

In this presentation we will look at a typical visual display environment in computer graphics to set the stage and provide a motivation for some of the things to be mentioned subsequently. There will be a brief, informal, intuitive review of the classical construction of B-splines, concentrating on simple knots, to provide a paradigm for some new developments. The use of B-splines in computer graphics to construct curves and surfaces is distinctly different from the use of B-splines in approximation and interpolation. This use will be presented, along with some of the reasons it is particularly appropriate from the point of view of computational efficiency and human interface. The work of Lyche, Riesenfeld, and Cohen for subdividing curves and surfaces, as a means of modification and display, will be outlined.

This will end one of the thrusts of the presentation. A second will cover the concept of geometric (as opposed to mathematical) continuity which was explored by Barsky, and which provides a generalisation of B-splines to functions which can serve as generators for "tensed" and "biased" splines. Some examples of curves produced by these functions, called Beta-splines, will be given.

The third portion of this presentation will cover recent progress in expanding the notion of geometric continuity to a more general context and in adapting the classical B-spline construction methods to the production of Beta-splines. The goal of this work is to develop subdivision recurrences for Beta-splines.

The primary references for the spline material in this presentation are [2,5,8] and [9]. Good backgrounds on the graphics environment are to be found in [1,6] and [7].

2. The Graphics Display Environment

The following is not the only example of a visual display environment for a design system, but it is typical and will serve to motivate the later discussion. Figure 1 gives an overview of a *display pipeline*. In it, mathematical objects are defined separately as *templates*, each in its own *local coordinate system*, each arranged according to some canonical scaling and orientation. These objects a placed together in a *world coordinate system* using rotations, translations and scalings, all of which are rigid transformations. As such, they preserve the character of the objects; in particular, polynomials remain polynomials. The result of these *modelling transformations* is a composite scene to be displayed. Each time a new view of the scene is to be taken, *viewing transformations* must be carried out. In industrial design systems these may be no more complicated than a single orthographic projection. In graphic art systems however, a camera model is often used: an eye-point is specified, as are viewing direction, upward orientation, angle of view, aperture, depth of field, and image plane. The *viewing frustum* which results is often mapped to a canonical viewing configuration by further rigid transformations. This canonical configuration is then subjected to a distortion by a *perspective transformation*. Magically, straight lines are preserved, but polynomials become rational functions. Under this distortion, the viewing frustum is changed into a canonical *clipping box*. Up until now, objects in the scene which are not within the viewing volume may have been involved in the computations. Algorithms are now invoked to trim away all portions of the scene outside of this volume and project only what remains onto the image plane. The projected objects must then be discretised, if the display is a *raster device*, or be approximated in outline by simple, known curves, if the display is a *calligraphic device*.

Profound implications are hidden in the above. To do all of this exactly, transformations would have to be applied to infinite numbers of points or to functional descriptions of surfaces; extensive root-finding techniques would be required to determine the curves of intersection between pairs of surfaces, as well as between surfaces and the sides of the viewing frustum; information would also have to be extracted to determine which surfaces are obscured by which – requiring that additional root-finding operations be performed to trace the silhouettes of objects with respect to the eye-point – and finally, discretisation processes akin to differential-equation solvers would have to be applied to paint an image on the display. For the purposes of industrial design, a wire-frame rendering of each object in the scene may suffice, but for other purposes a more realistic rendering of the objects may be required. This may involve something as elaborate as using a mathematical model of the optical characteristics of some collection of materials together with computations requiring normal vectors to the various surfaces and the illuminant details of a number of light sources. If all of this is to be interactive, then it must take place as fast as the refresh rate of the display – that is: typically within a sixtieth to a thirtieth of a second. (It is small wonder that the Cray computer corporation has started to advertise in the graphics trade literature.) Finally, if machine-tool descriptions are needed, while they don't need to be performed with the speed of display computations, they will still involve determining how to track contours on a surface while holding a prescribed orientation to the normal – which could be a nontrivial problem in control theory.

The only salvation for owners of small computers lies in determining how little one can do, how efficiently and how approximately, without the disturbing the final result within visual or machine-tool tolerances. In graphics this is accomplished by "creative cheating", which frequently involves applying all of the above processes only to a small numbers of representers of the objects in question. For example, it is still usual to deal mostly with polyhedral bodies, plus a few additional primitive forms such as spheres and cylinders, and carry out the transformations of the pipeline merely on vertices, centres, radii, etc., using the images of these few representers at the bottom of the pipeline to recreate approximate pictures. One feature of the unique way in which spline surfaces are used in graphics derives from the fact it provides an efficient way for the surfaces to be approximated to arbitrary accuracy by polyhedral surfaces. It is to these polyhedra rather than to the splines themselves that the above computations in the pipeline can be applied.

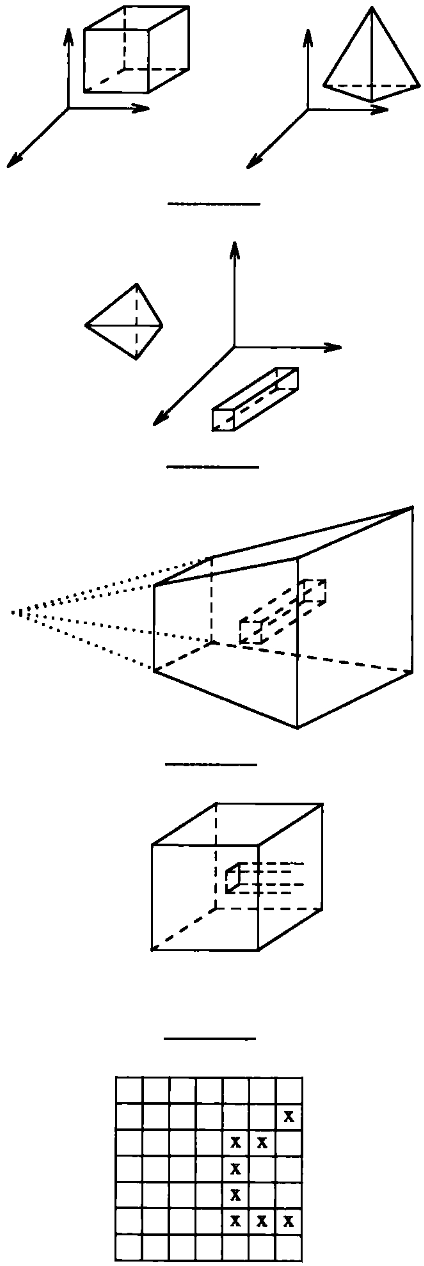

Figure 1. The graphics display pipeline.

3. Splines

For any $k > 0$, let $\mathbf{P}^k$ be the space of polynomials of order k with real coefficients:

$$p(u) \;=\; c_0 + c_1 u + \cdots + c_{k-1} u^{k-1} \;.$$

For any $m > 0$ let $\mathbf{U}^m$ be a sequence of m *knots*:

$$\mathbf{U}^m \;=\; \{ u_0, \ldots, u_m \} \;,$$

where

$$u_0 \le u_1 \le \cdots \le u_m \;.$$

Each knot u_i has *multiplicity* μ_i, the count of the knots in $\mathbf{U}^m$ having value equal to u_i. This includes u_i itself, By convention, $\mu_i \le k$ for all i; hence, $u_i < u_{i+k}$ for all $i = 0, \ldots, m-k$. As a further convention, let $a < u_0$ and $b > u_m$, and agree that nothing outside of $[a, b)$ is of any interest.

The set of the M *distinct*, consecutive values in $\mathbf{U}^m$

$$u_{i_0} < \cdots < u_{i_M}$$

are the *breakpoints* of $\mathbf{U}^m$. ($\{ i_0, \ldots, i_m \} \subseteq \{ 0, \ldots, m \}$ is any conveniently chosen subsequence which picks out the breakpoints.) Each breakpoint is associated with the multiplicity of its corresponding knots, and the *breakpoint intervals* defined by $\mathbf{U}^m \bigcup [a, b)$ are the half-open intervals

$$\mathbf{I}_0 \;=\; [a, u_{i_0})$$

$$\mathbf{I}_j \;=\; [u_{i_{j-1}}, u_{i_j}) \quad \text{for } j = 1, \ldots, M \;.$$

and

$$\mathbf{I}_{M+1} \;=\; [u_{i_M}, b) \;.$$

Formally:

Definition: Assuming that $k \ge 1$, that $m \ge k-1$, and that $1 \le \mu_i \le k$ for all $i = 0, \ldots, m$, then $\mathbf{S}(\mathbf{P}^k, \mathbf{U}^m, a, b)$, the set of all splines of order k on $[a,b)$ with the knot sequence $\mathbf{U}^m \subset [a,b)$, breakpoints $\{ u_{i_0}, u_{i_1}, \ldots, u_{i_M} \} \subseteq \mathbf{U}^m$, and breakpoint intervals $\mathbf{I}_0, \ldots, \mathbf{I}_{M+1}$ is the set of all functions of the form $s(u)$ satisfying:

$$s(u) \in \mathbf{P}^k \;\text{ for each } \mathbf{I}_j \; (j = 0, \ldots, M+1)$$

and for any breakpoint u_{i_j} associated with multiplicity μ_{i_j} $(j = 0, \ldots, M)$, if

$$s(u) \equiv p_j \in \mathbf{P}^k \quad \text{on } \mathbf{I}_j$$

and

$$s(u) \equiv p_{j+1} \in \mathbf{P}^k \quad \text{on } \mathbf{I}_{j+1}$$

then

$$D_u^{(l)} p_j(u_{i_j}) \;=\; D_u^{(l)} p_{j+1}(u_{i_j}) \;\text{ for } l = 0, \ldots, k-1-\mu_{i_j} \;.$$

In the definition, $D_u^{(l)} p(u_i)$ stands for the l^{th} derivative with respect to u of $p(u)$ evaluated at u_i. We denote p_j, p_{j+1} as the *segment polynomials* which describe $s(u)$ in the interval $\mathbf{I}_j$ and $\mathbf{I}_{j+1}$ respectively. Notice that any issue of end conditions is left open. It is only necessary that a spline be a polynomial in the intervals $\mathbf{I}_0$ and $\mathbf{I}_{M+1}$. Imposing various end conditions will serve merely to isolate subsets of splines, and this issue will be left aside.

It is easily verified that $\mathbf{S}(\mathbf{P}^k, \mathbf{U}^m, a, b)$ is a vector space with $\mathbf{P}^k$ as a subspace.

4. The One-Sided Basis

The dimension of $\mathbf{S}(\mathbf{P}^k, \mathbf{U}^m, a, b)$ is

$$k + \mu_{i_0} + \cdots + \mu_{i_M} .$$

This can be made plausible by considering any $s(u) \in \mathbf{S}(\mathbf{P}^k, \mathbf{U}^m, a, b)$ as u moves from a rightwards to b. On the breakpoint interval $\mathbf{I}_0 = [a, u_{i_0})$, $s(u) = p_0(u)$ is a polynomial of order k; hence it can be represented as a linear combination of

$$(u-a)^0, (u-a)^1, \ldots, (u-a)^{k-1} . \qquad (4.1)$$

In the next interval to the right $u_{i_0} \le u < u_{i_1}$, $s(u)$ changes to

$$\begin{aligned} s(u) &= p_0(u) + (p_1(u) - p_0(u)) \ . \\ &= p_0(u) + \Delta_{1_1}(u) \ , \end{aligned}$$

where $\Delta_1(u)$ "touches zero with $C^{k-1-\mu_{i_0}}$ continuity" at $u = u_{i_0}$. It proves true that $\Delta_1(u)$ can be represented as a linear combination of the truncated power functions

$$(u-u_{i_0})_+^{k-1}, (u-u_{i_0})_+^{k-2}, \ldots, (u-u_{i_0})_+^{k-\mu_{i_0}} . \qquad (4.2)$$

Consequently $s(u)$ can be represented on the interval $a \le u < u_{i_1}$ as a linear combination of the functions (4.1) together with those of (4.2).

The same arguments apply as u crosses u_{i_j} and $p_j(u)$ changes into

$$p_{j+1}(u) = p_j(u) + \Delta_{j+1}(u)$$

for each $j = 0, \ldots, M$.

After further considerations:

Theorem: The functions

$$(u-a)^0, (u-a)^1, \ldots, (u-a)^{k-1} ,$$

together with

$$\begin{array}{lr} (u-u_{i_0})_+^{k-1}, (u-u_{i_0})_+^{k-2}, \ldots, (u-u_{i_0})_+^{k-\mu_{i_0}} & \text{for } u_{i_0} \\ (u-u_{i_1})_+^{k-1}, (u-u_{i_1})_+^{k-2}, \ldots, (u-u_{i_1})_+^{k-\mu_{i_1}} & \text{for } u_{i_1} \\ \bullet\bullet\bullet \qquad \bullet\bullet\bullet & \bullet\bullet\bullet \\ (u-u_{i_M})_+^{k-1}, (u-u_{i_M})_+^{k-2}, \ldots, (u-u_{i_M})_+^{k-\mu_{i_M}} & \text{for } u_{i_M} \end{array} \qquad (4.3)$$

form a basis for $\mathbf{S}(\mathbf{P}^k, \mathbf{U}^m, a, b)$.

There are precisely $k + \mu_{i_0} + \cdots + \mu_{i_M}$ functions in (4.3). This is the *one-sided basis* of $\mathbf{S}(\mathbf{P}^k, \mathbf{U}^m, a, b)$.

5. Linear Combinations and Cancellation

Computing the coefficients in the representation of a spline with respect to this basis is often an ill-conditioned problem. This arises, roughly, as follows. Most splines with which we would want to deal in practice have moderate values throughout the interval $[a,b)$. The one-sided basis functions, on the other hand, blow up as u increases. Hence, if this basis is used to express "reasonable" spline curves and surfaces, the coefficients required for this could be expected to flip-flop between large positive and negative values in order to force *numerical cancellation* of the basis-function values as u increases.

A second shortcoming, from the point of view of graphics, is that the one-sided basis functions do not have *compact support*; they are all nonzero on half of the real line. If a curve or surface is represented in terms of the one-sided basis and some change is made to the representation to provide an adjustment of shape, then the change has an influence over the entire curve or surface. A complete recomputation of the curve or surface is necessary; no local updates are possible. The continual need engage in costly recomputations will all but rule out interactive graphical design. In graphics it is as significant that the B-splines have compact support as that they provide well-conditioned representations.

The key to constructing a desirable basis from the less desirable (but conceptually simple) one-sided basis is to recognise (1) that a basis with compact support will be an answer to the numerical objections above as well as (2) being desirable from the point of view of computational efficiency. Compact support can be achieved by a process of *symbolic cancellation*, before any numerical computations are begun. It is through this door that divided differences enter.

To illustrate, let $k = 4$ (cubic splines), and consider

$$u_i < u_{i+1} < u_{i+2} < u_{i+3} < u_{i+4} \ .$$

We have

$$\frac{(u-u_{i+1})_+^3 - (u-u_i)_+^3}{(u_{i+1}-u_i)} = \begin{cases} 0 & u < u_i \\ -(u-u_i)^3 & u_i \le u < u_{i+1} \\ -3u^2 + 3u(u_{i+1}+u_i) & u_{i+1} \le u \ . \\ \quad -(u_{i+1}^2 + u_{i+1}u_i + u_i^2) & \end{cases}$$

which is a "nicer" function than either $(u-u_i)_+^3$ or $(u-u_{i+1})_+^3$ in that it grows only quadratically for $u \to \infty$. Denote the result by

$$[u_i, u_{i+1}:t](u-t)_+^3 = \frac{(u-u_{i+1})_+^3 - (u-u_i)_+^3}{u_{i+1} - u_i} \ .$$

Since this function is a linear combination of $(u-u_i)_+^3$ and $(u-u_{i+1})_+^3$, we may substitute it for one of these truncated power functions, e.g. for $(u-u_i)_+^3$. We may carry out a similar operation for the pairs

$$\{u_{i+1}, u_{i+2}\}\ ,\ \{u_{i+2}, u_{i+3}\}\ ,\ \text{and}\ \{u_{i+3}, u_{i+4}\}\ ,$$

to produce quadratic-growing substitutes for

$$(u-u_{i+1})_+^3\ ,\ (u-u_{i+2})_+^3\ ,\ \text{and}\ (u-u_{i+3})_+^3\ .$$

Going one stage further, let

$$[u_i, u_{i+1}, u_{i+2}:t](u-t)_+^3 = \frac{[u_{i+1}, u_{i+2}:t](u-t)_+^3 - [u_i, u_{i+1}:t](u-t)_+^3}{u_{i+2} - u_i}\ ,$$

This function grows only linearly for $u \to \infty$. It may be used to replace $[u_i, u_{i+1}:t](u-t)_+^3$, and the differencing and replacement may be repeated pairwise, again. Ultimately we arrive at a function which "grows as zero" for $u \to \infty$.

When multiple knots appear, the construct

$$[u_i, u_{i+1}:t](u-t)_+^3 = \frac{(u-u_{i+1})_+^3 - (u-u_i)_+^3}{u_{i+1} - u_i}$$

is regarded as being taken in the limit as $u_i \to u_{i+1}$. This provides the convention that the derivatives of