CALCULUS
Theory and Applications

Volume I

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CALCULUS
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Volume I

Preface

Calculus consists of the study of limits of various sorts and the systematic exploitation of the completeness axiom. It was developed over a period of several hundred years in order to solve problems from the physical sciences. It is the language by which precision and quantitative predictions for many complicated problems are obtained. It also has significant applications in pure mathematics. For example, it is used to define and find lengths of curves, areas and volumes of three dimensional shapes. It is essential in order to solve many maximization problems and it is prerequisite material in order to understand models based on differential equations. These and other applications are discussed to some extent in this book.

It is assumed the reader has a good understanding of algebra on the level of college algebra or what used to be called algebra II along with some exposure to geometry and trigonometry although the book does contain a review of these things.

If the optional sections and nonstandard sections are not included, this book is fairly short. However, there is a lot of nonstandard material, including the big theorems of advanced calculus.

I have tried to give complete proofs of all theorems somewhere in the book because I believe that calculus is part of mathematics and that in mathematics the validity of some assertion is typically established by giving a proof. This is certainly true in algebra so it seems to me it should also be true in calculus. Mathematics is not about accepting on faith unproved assertions presumably understood by someone else.

I expect the reader to be able to use a calculator whenever it would be helpful to do so. In addition, a minimal introduction to the use of computer algebra systems is presented. Having said this, calculus is not about using calculators or any other form of technology. Weierstrass could do calculus quite well without the benefit of modern gadgets. I believe that when the syntax and arcane notation associated with technology are presented too prominently, these things become the topic of study rather than the concepts of calculus and this is a real shame. This is a book on calculus.

Pictures are often helpful in seeing what is going on, and there are many pictures in this book for this reason. However, calculus is not art and ultimately rests on logic and definitions, like the rest of mathematics. Algebra plays a central role in gaining the sort of understanding which generalizes to higher dimensions where pictures are not available.

A star next to an exercise indicates it is either technically difficult or perhaps a little bit challenging to figure out. Do not be frightened by these exercises. They just might take a little longer to work but some are very worth while and ought to be attempted.

Supplementary material for this text including routine exercise sets is available at http://www.math.byu.edu/~klkuttle/CalculusMaterials. I have made these exercise sets using Scientific Workplace Exam Builder. The source file is available at this site and you can modify it to include more types of exercises if you desire.

The topics discussed in this book are arranged in a typical order for most calculus courses I have seen, with the notable exception of the early introduction of sequences and their limits. The concepts of limit of a function of a real variable and limit of a sequence were made rigorous at around the same time in the nineteenth century and also, a sequence is a type of function, so I think this order makes good sense. Another advantage is that continuity and various theorems about continuous functions can be understood by some people, more easily in terms of convergent sequences than in terms of the traditional epsilon delta definition.

Instead of the order listed in the table of contents, one could begin with the chapter on Page 323 and do all the topics about vectors in this and the remaining chapters of the book before beginning the usual topics of one variable calculus in the chapter which begins on Page 39. Just leave out the material on the parabolic mirror and an occasional exercise which depend on the derivative. This approach may be better because students often encounter these topics in their physics classes at the time they begin calculus.

I am grateful to World Scientific for publishing this and the second volume of this calculus book. I am also grateful to Kate Phillips for the many drawings which I could not have done.

Contents

Chapter 1

A Short Review Of Precalculus

1.1 Set Notation

A set is just a collection of things called elements. Often these are also referred to as points in calculus. For example $\{1,2,3,8\}$ would be a set consisting of the elements 1,2,3, and 8. To indicate that 3 is an element of $\{1,2,3,8\}$, it is customary to write $3 \in \{1,2,3,8\}$. $9 \notin \{1,2,3,8\}$ means 9 is not an element of $\{1,2,3,8\}$. Sometimes a rule specifies a set. For example you could specify a set as all integers larger than 2. This would be written as $S = \{x \in \mathbb{Z} : x > 2\}$. This notation says: the set of all integers x, such that $x > 2$.

If A and B are sets with the property that every element of A is an element of B, then A is a subset of B. For example, $\{1,2,3,8\}$ is a subset of $\{1,2,3,4,5,8\}$, in symbols, $\{1,2,3,8\} \subseteq \{1,2,3,4,5,8\}$. The same statement about the two sets may also be written as $\{1,2,3,4,5,8\} \supseteq \{1,2,3,8\}$.

The union of two sets is the set consisting of everything which is contained in at least one of the sets A or B. As an example of the union of two sets, $\{1,2,3,8\} \cup \{3,4,7,8\} = \{1,2,3,4,7,8\}$ because these numbers are those which are in at least one of the two sets. In general

$$A \cup B \equiv \{x : x \in A \text{ or } x \in B\}.$$

Be sure you understand that something which is in both A and B is in the union. It is not an exclusive or.

The intersection of two sets A and B consists of everything which is in both of the sets. Thus $\{1,2,3,8\} \cap \{3,4,7,8\} = \{3,8\}$ because 3 and 8 are those elements the two sets have in common. In general,

$$A \cap B \equiv \{x : x \in A \text{ and } x \in B\}.$$

The symbol $[a,b]$ denotes the set of real numbers x, such that $a \leq x \leq b$ and $[a,b)$ denotes the set of real numbers such that $a \leq x < b$. (a,b) consists of the set of real numbers x such that $a < x < b$ and $(a,b]$ indicates the set of numbers x such that $a < x \leq b$. $[a,\infty)$ means the set of all numbers x such that $x \geq a$ and $(-\infty,a]$ means the set of all real numbers which are less than or equal to a. These sorts of sets of real numbers are called intervals. The two points a and b are called endpoints of the interval. Other intervals such as $(-\infty, b)$ are defined by analogy to what was just explained. In general, the curved parenthesis indicates the end point it sits next to is not included while the square parenthesis indicates this end point is included. The reason that there will always be a curved parenthesis next to ∞ or $-\infty$ is that these are not real numbers. Therefore, they cannot be included in any set of real numbers. The symbol is called "infinity" or minus "infinity".

A special set which needs to be given a name is the empty set also called the null set, denoted by $\emptyset$. Thus $\emptyset$ is defined as the set which has no elements in it. Mathematicians like to say the empty set is a subset of every set. The reason they say this is that if it were not so, there would have to exist a set A, such that $\emptyset$ has something in it which is not in A. However, $\emptyset$ has nothing in it and so the least intellectual discomfort is achieved by saying $\emptyset \subseteq A$.

If A and B are two sets, $A \setminus B$ denotes the set of things which are in A but not in B. Thus

$$A \setminus B \equiv \{x \in A : x \notin B\}.$$

Set notation is used whenever convenient.

1.2 Completeness Of The Real Numbers

I assume the reader is familiar with the usual algebraic properties of the real numbers. However, they have another property known as completeness.

Definition 1.1. A nonempty set $S \subseteq \mathbb{R}$ is bounded above (below) if there exists $x \in \mathbb{R}$ such that $x \geq (\leq) s$ for all $s \in S$. If S is a nonempty set in $\mathbb{R}$ which is bounded above, then a number l which has the property that l is an upper bound and that every other upper bound is no smaller than l is called a least upper bound, $l.u.b.(S)$ or often $\sup(S)$. If S is a nonempty set bounded below, define the greatest lower bound, $g.l.b.(S)$ or $\inf(S)$ similarly. Thus g is the $g.l.b.(S)$ means g is a lower bound for S and it is the largest of all lower bounds. If S is a nonempty subset of $\mathbb{R}$ which is not bounded above, this information is expressed by saying $\sup(S) = +\infty$ and if S is not bounded below, $\inf(S) = -\infty$.

Every existence theorem in calculus depends on some form of the completeness axiom.

Axiom 1.1. *(completeness) Every nonempty set of real numbers which is bounded above has a least upper bound and every nonempty set of real numbers which is bounded below has a greatest lower bound.*

It is this axiom which distinguishes Calculus from Algebra.

A fundamental result about sup and inf is the following.

Proposition 1.1. Let S be a nonempty set and suppose $\sup(S)$ exists. Then for every $\delta > 0$,

$$S \cap (\sup(S) - \delta, \sup(S)] \neq \emptyset.$$

If $\inf(S)$ exists, then for every $\delta > 0$,

$$S \cap [\inf(S), \inf(S) + \delta) \neq \emptyset.$$

Proof: Consider the first claim. If the indicated set equals $\emptyset$, then $\sup(S) - \delta$ is an upper bound for S which is smaller than $\sup(S)$, contrary to the definition of $\sup(S)$ as the least upper bound. In the second claim, if the indicated set equals $\emptyset$, then $\inf(S) + \delta$ would be a lower bound which is larger than $\inf(S)$ contrary to the definition of $\inf(S)$. ■

1.3 A Few Algebraic Conventions And Techniques

Summation notation is a convenient way to specify a sum.

Definition 1.2. For $i = m, \cdots, n$ let a_i be specified. Then

$$\sum_{i=m}^{n} a_i \equiv a_m + a_{m+1} + \cdots + a_n$$

I will use this whenever convenient.

Example 1.1. Find $\sum_{i=1}^{3} 2i - 1$.

From the definition, it equals $(2-1) + (2 \times 2 - 1) + (2 \times 3 - 1) = 9$.

An important technique is the technique of proof by **induction**. I will illustrate with a simple example which is useful for its own sake.

Example 1.2. For $n = 1, 2, 3, \cdots$ and $\alpha > 0$ it is always the case that

$$(1+\alpha)^n \geq 1 + n\alpha + \frac{n(n-1)}{2}\alpha^2$$

Here is why. The statement is true if $n = 1$. Now suppose I can show that whenever the statement is true for some value of n it follows that it must be true for the next value of n. Then it must be the case that it is true for each value of n. Consider why this is. Since it is true for $n = 1$, and whenever it is true for some n, it is true for $n + 1$, it follows that it must be true for 2. Since it is true for 2, it must be true for 3 by the same reasoning, and so forth. Thus it suffices to show that **if** it is true for n **then** it is true for $n + 1$. I haven't done this yet but I am about to do it.

Suppose then that the inequality is true for n. I need to verify that with this assumption, it holds for $n+1$. That is, the same formula needs to hold with n replaced everywhere with $n+1$.

Using the assumption that it is true for n,

$$\begin{aligned}(1+\alpha)^{n+1} &= (1+\alpha)(1+\alpha)^n \\ &\geq (1+\alpha)\left(1+n\alpha+\frac{n(n-1)}{2}\alpha^2\right) \\ &= 1+n\alpha+\frac{n(n-1)}{2}\alpha^2+\alpha+n\alpha^2+\frac{n(n-1)}{2}\alpha^3 \\ &\geq 1+(n+1)\alpha+\frac{n(n-1)}{2}\alpha^2+n\alpha^2\end{aligned}$$

where I simply threw out the last term in going to the last line. This equals

$$\begin{aligned}&1+(n+1)\alpha+\frac{n(n-1)+2n}{2}\alpha^2 \\ &= 1+(n+1)\alpha+\frac{(n+1)n}{2}\alpha^2\end{aligned}$$

Thus the inequality holds with n replaced with $n+1$. This proves the desired inequality.

There are many other examples where induction is useful.

Also of use is the concept of **absolute value** of a number. This is defined as follows.

$$|x| \equiv \text{ the distance from } x \text{ to } 0 \text{ on the number line.}$$

An equivalent way of defining it is to say $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

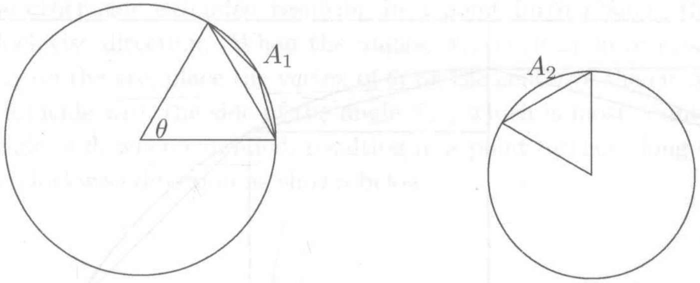
1.4 The Circular Arc Subtended By An Angle

How can angles be measured? This will be done by considering arcs on a circle. To see how this will be done, let θ denote an angle and place the vertex of this angle at the center of the circle. Next, extend its two sides till they intersect the circle. Note the angle could be opening in any of infinitely many different directions. Thus this procedure could yield any of infinitely many different circular arcs. Each of these arcs is said to **subtend** the angle. In fact each of these arcs has the same length. When this has been shown, it will be easy to measure angles. Angles will be measured in terms of lengths of arcs subtended by the angle. Of course it is also necessary to define what is meant by the length of a circular arc in order to do any of this. First I will describe an intuitive way of thinking about this and then give a rigorous definition and proof. If the intuitive way of thinking about this satisfies you, no harm will be done by skipping the more technical discussion which follows.

Take an angle and place its **vertex** (the point) at the center of a circle of radius r. Then, extending the sides of the angle if necessary till they intersect the circle, this determines an arc on the circle. If r were changed to R, this really amounts to a change of units of length. Think for example, of keeping the numbers the same but changing centimeters to meters in order to produce an enlarged version of the same picture. Thus the picture looks exactly the same, only larger. It is reasonable to suppose, based on this reasoning, that the way to measure the angle is to take the length of the arc subtended in whatever units being used, and divide this length by the radius measured in the same units, thus obtaining a number which is independent of the units of length used, just as the angle itself is independent of units of length. After all, it is the same angle regardless of how far its sides are extended. This is in fact how to define the radian measure of an angle, and the definition is well defined. Thus in particular, the ratio between the circumference (length) of a circle and its radius is a constant which is independent of the radius of the circle[1]. Since the time of Euler in the 1700's, this constant has been denoted by 2π. In summary, if θ is the radian measure of an angle, the length of the arc subtended by the angle on a circle of radius r is $r\theta$.

This is a little sloppy right now because no precise definition of the length of an arc of a circle has been given. For now, imagine taking a string, placing one end of it on one end of the circular arc and then wrapping the string till you reach the other end of the arc. Stretching this string out and measuring it would then give you the length of the arc. Such intuitive discussions involving string may or may not be enough to convey understanding. If you need to see more discussion, read on. Otherwise, skip to the next section.

To give a precise description of what is meant by the length of an arc, consider the following picture.



In this picture, there are two circles, a big one having radius R and a little one having radius r. The angle θ is situated in two different ways subtending the arcs

[1]In 2 Chronicles 4:2 the "molten sea" found in Solomon's temple is described. It sat on 12 oxen, was round, 5 cubits high, 10 across and 30 around. Thus its radius was 5 and the Bible, taken literally, gives the value of π as 3. This is not too far off but is not correct. Other incorrect values of π can be found in the Indiana pi bill of 1897. Later, methods will be given which allow one to calculate π more precisely. A better value is 3.1415926535 and presently this number is known to thousands of decimal places. It was proved by Lindeman in 1882 that π is transcendental which is the worst sort of irrational number.